HEAT CONDUCTION WITH MAPLE

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PREFACE

The theory of heat conduction has been an integral part of heat transfer teaching and research for over a century. In typical undergraduate heat transfer courses, the treatment of heat conduction is usually limited to the basic theory and elementary applications. The more advanced material is reserved for presentation in advanced heat transfer courses or courses that are exclusively devoted to heat conduction. Such courses are almost always devoted to the teaching of classical mathematical techniques, such as the methods of separation of variables, Laplace and Fourier transforms, integral methods, perturbation theory and Green functions. Increasingly, numerical techniques such finite differences and finite elements are being incorporated to complement the discussion of the classical methods. This presentation methodology is reflected in several excellent textbooks on heat conduction that are currently available.

The learning and implementation of classical mathematical techniques entails tedious algebraic manipulations. While they illuminate the elegance of mathematics, they often take the focus away from understanding the physics of the analysis. Furthermore, even if the problem is amenable to an analytical solution, the algebra is often too intense or laborious. This difficulty can now be surmounted with the use of symbolic algebra packages such as Maple or Mathematica. These programs are virtually revolutionizing the teaching of calculus and differential equations. However, their use in teaching and research in the traditional disciplines of engineering is still in its infancy.

The aim of the present book is to demonstrate how the use of Maple not only facilitates the study of traditional topics in heat conduction theory but also opens the door for introducing more creative and challenging problems at both undergraduate and graduate levels. The choice of Maple in favour of Mathematica is a personal one and no way reflects the superiority of Maple over Mathematica. Besides its symbolic analytic capability, Maple also has powerful numerical and graphical interfaces which make the software very versatile. This book was written with Maple 8 but the material should easily transition, often without any modification, into any future version of the software.

The book can be used either as a self-contained study of heat conduction theory or as a supplement to a standard heat conduction textbook. In either case, an undergraduate background in heat transfer and differential equations would be helpful. The material has been class tested for several years and proven effective in enhancing student's learning of heat transfer. The students found that the learning of Maple demanded only a modest amount of effort but the broadened their analytical ability considerably. Most of them also enjoyed using Maple in other engineering courses.

The opening chapter of the book, which is devoted to an overview of Maple, is designed to familiarize the reader with the basic syntax and features of the software. For an in-depth presentation of Maple, the reader may refer to the references that are cited throughout this book. The second chapter illustrates the use of Maple to study and evaluate the mathematical functions that are commonly encountered in heat conduction analysis. Here the emphasis is on error functions, gamma and beta functions, Bessel functions, Legendre functions, hypergeometric functions and the exponential integral function. Chapter 3 shows how Maple can be used to solve algebraic and transcendental equations that arise in heat conduction problems. The study of elementary one-dimensional steady state conduction with Maple is presented in chapter 4. The topics include conduction in plane wall, hollow cylinder, hollow sphere and truncated conical section. The chapter also discusses heat conduction with uniform heat generation. The more advanced topics in one-dimensional steady conduction, such as variable thermal conductivity, non uniform heat generation, radiative-convective boundary condition and optimization of thermal systems are covered in chapter 5. A comprehensive discussion of the analysis and design of extended surfaces appears in chapter 6. Included are straight fins, spines, annular fins, finned arrays, and convecting-radiating fins. Chapter 7 focuses on two-dimensional steady conduction. Here the method of separation of variables and the finite difference method are implemented in the Maple environment for Cartesian, cylindrical, and spherical geometries. The last chapter treats the topic of time dependent conduction. The presentation begins with lumped thermal capacity models with constant properties. Temperature dependent specific heat, radiative cooling and the temperature dependent heat transfer coefficient are analyzed with the help of Maple. Time-dependent conduction in a semi-infinite solid is studied with the method of the Laplace transformation and the similarity technique, both of which are implemented in Maple. Solutions for the transient response of an infinitely long plane wall and an infinitely long solid cylinder, both experiencing surface convection, are then considered. Here the opportunity is taken to illustrate the implementation of the method of separation of variables, the explicit finite difference method, and the implicit finite difference method. Finally, the method of complex combination is used to study the oscillatory behaviour of a convecting rectangular fin with periodically varying base temperature.

The book is replete with examples that range from elementary to advanced. Some of them are chosen from the contemporary heat transfer literature while others are novel extensions of known problems. A careful study of these examples should readily suggest and motivate the reader to exploit Maple to explore new and challenging problems for either individual research or as assigned in undergraduate and graduate heat conduction courses.

The author would like to express his gratitude to several people who have influenced the writing of this book. My wife, Dr. Ayesha Aziz always supports and appreciates my writing endeavours. Our children, Fahad, Sheza, and Kashif and their spouses Sana, Sameer, and Tania provided technical and moral support during this project. Our two grandchildren, Senaan and Fiza, were always there to relieve my stress with their beautiful smiles and funny antics. Professor Robert Lopez, formerly at Rose-Hulman Institute of Technology and now at Maplesoft, taught me Maple many years ago and who was kind enough to critique the first draft of this book. My former student, Greg McFadden, assisted with the typing of the manuscript.

This book is dedicated to Dr. Ayesha Aziz as a gift of love.

A. Aziz

Chapter 1

An Overview of Maple

1.1 INTRODUCTION

The use of symbolic codes such as *Maple, Mathematica*, *Macsyma* and others for engineering computation has grown steadily in recent years. While programming languages such as *FORTRAN, Basic* and *C* are designed for numerical computations, symbolic codes provide a rich variety of symbolic, graphical and numerical capabilities in a single code. With these codes, one can use the digital computer to carry out algebraic and trigonometric operations, differentiation and integration, series expansions, matrix operations, Laplace transformations, and the solution (analytical and numerical) of ordinary and partial differential equations. While currently available symbolic codes can handle only a limited number of solution methods for partial differential equations, a symbolic code can be used to develop the solutions for a number of problems associated with partial differential equations [1].[1] When a finite difference or finite element approach is used in conjunction with a symbolic language, the combination becomes a powerful tool for engineering analysis [2].

Traditionally, heat conduction theory has been taught and practiced using classical mathematical tools. Techniques such as the separation of variables, Green's function, the Laplace transformation, integral transforms and others have been standard procedure for generations. However, fruitful pursuit of these techniques entails the use of tedious and often painstaking algebraic manipulations. Today's symbolic codes offer much needed relief and open the door for the convenient and efficient exploration of the fascinating realm of heat conduction theory.

[1] Numbers in brackets designate references to be found at the end of the chapter.

1.2 AN OVERVIEW OF MAPLE

Maple is one of several computer algebra systems that are commonly available. The system was developed by a group of teachers and researchers at the University of Waterloo in Waterloo, Ontario in the 1980's. Since its inception, the code has been extended and enhanced many times. Both student and professional editions of the program are available. An excellent introduction to *Maple* is given by Heck [3].

1.2.1 Statements and Commands

Like other symbolic codes, *Maple* is an interactive tool that allows the user to enter commands which, in turn, cause the computer to respond. When the program is activated, a prompt in the form of a > is displayed. *Maple* has hundreds of built-in commands. The commands that the user typically enters are algebraic expressions, assignments or function calls. The case sensitive nature of *Maple* must be kept in mind when entering commands because most *Maple* commands employ lower case letters. Every statement executed by *Maple* must be terminated by a semicolon (;) or a colon (:). The semicolon prompts *Maple* to provide the result of statement execution whereas a colon suppresses the display of the result. A statement may be extended over several lines and it is also possible to enter multiple commands on a single line or to combine several commands in a single statement.

On occasions, it is convenient to refer to an immediately preceding result. This can be done with the percent symbol (%). Double percent (%%) and triple percent (%%%) refer to the second to last and third to last results respectively. Caution should be exercised when using the previous statement command as *Maple* calls the previous statement that remains in memory which is not necessarily the last command shown on the input window. Good programming practice requires one to re-compute the entire worksheet to ensure proper functioning of the previous statement command.

1.2.2 Elementary Operations

Maple does simple arithmetic operations both symbolically and numerically. The symbols +, –, *, /, ^ or **, !, **sqrt**, and **abs** instruct *Maple* to add, subtract, multiply, divide, exponentiate, take factorials, take square roots and take absolute values, respectively. *Maple* follows the same order of precedence as other programming languages such as *FORTRAN* or *C*. However, parentheses can and should be used to ensure the desired order of operational or functional precedence.

Maple has many built in constants and functions. The capital letter I denotes $\sqrt{-1}$. The symbol Pi is recognized as the mathematical constant, 3.14159... (displayed simply as π) whereas pi is interpreted as the Greek letter π. A similar distinction must be kept in mind when using γ; *Maple* uses gamma (in input) and gamma (in output) for the Euler constant. The constant ∞ has the name infinity.

Example 1.1
This example provides a sample of *Maple* statements and commands that were described in sections 1.2.1 and 1.2.2. It includes examples of algebraic expressions, elementary operations, use of percentage symbols (single and double), and commands Pi, gamma, and infinity.

```
> a*x^2+b*x*c;
```

$$a\,x^2+b\,x\,c$$

```
> y:=9*x^3-21*x^2+5*x+23;
```

$$y := 9\,x^3-21\,x^2+5\,x+23$$

```
> y;
```

$$9\,x^3-21\,x^2+5\,x+23$$

```
> %^2;
```

$$(9\,x^3-21\,x^2+5\,x+23)^2$$

```
> sqrt(x);
```

$$\sqrt{x}$$

```
> sqrt(3+4*I);
```

$$2+I$$

```
> abs(x^2);
```

$$|x|^2$$

```
> pi;
```

$$\pi$$

```
> Pi;
```

$$\pi$$

Note that both Pi and pi are shown symbolically, however Pi is considered the numerical constant and pi is merely the Greek variable pi.

```
> gamma;
```

$$\gamma$$

```
> infinity;
```

$$\infty$$

Maple has the ability to manipulate (symbolically) and evaluate many mathematical functions. Table 1.1 lists the functions that are commonly used in heat conduction theory and applications.

Table 1.1 Mathematical Functions

Mathematical Function	**MAPLE syntax**
Exponential function	**exp**
Natural logarithm	**1n, log**
Trigonometric functions	**sin, cos, tan cot, sec, csc**
Hyperbolic functions	**sinh, cosh, tanh coth, sech, csch,**
Error function	**erf**
Complimentary error function	**erfc**
n^{th} repeated integral of complementary error function	**erfc(n,x)**
Bessel functions	**Bessel J, Bessel Y**
Modified Bessel functions	**Bessel I, Bessel K**
Gamma function	**GAMMA**
Airy wave functions	**AiryAi, AiryBi**
Legendre polynomial	**P**
Laplace transform	**laplace,**
Inverse Laplace transform	**invlaplace**
Generalized hypergeometric function	**hypergeom**
Beta Function	**Beta**
Heaviside step function	**Heaviside**
Dirac delta function	**Dirac**
Exponential integral	**Ei**

Example 1.2
This example illustrates the evaluation of various mathematical functions identified in Table 1.1. Note that the commands for Legendre polynomial, **P(n,x)**, Laplace transform, **laplace**, inverse Laplace transform, **invlaplace**, and hypergeometric function, **hypergeom**, are contained in their respective *Maple* libraries (see section 1.3) and these libraries must be loaded before *Maple* can evaluate these functions. For example, the library orthopoly is loaded with the command **with(orthopoly)** to obtain the fifth order Legendre polynomial **P(5,x)**. Notice the use of the **simplify** command to force the symbolic evaluation and display of the **hypergeom** function.

```
> exp(0.2);
```

$$1.221402758$$

```
> ln(12.8);
```

$$2.549445171$$

```
> sin(Pi/2);
```

$$1$$

```
> tanh(0.5);
```

$$0.4621171573$$

```
> erfc(0.2);
```

$$0.7772974108$$

```
> erfc(2,0.5);
```

$$0.06996472345$$

```
> BesselI(1/3,0.30);
```

$$0.6050965185$$

```
> GAMMA(3.6);
```

$$3.717023853$$

```
> AiryAi(1.45);
```

$$0.07675466713$$

```
> AiryBi(-2.6+3.1*I);
```

$$-28.13276864 + 25.66995337\, I$$

```
> with(orthopoly):
> P(5,x);
```

$$\frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8}x$$

```
> with(inttrans):
```

```
> laplace(BesselJ(0,k*t),t,s);
```

$$\frac{1}{\sqrt{s^2+k^2}}$$

```
> invlaplace(1/sqrt(s^2+k^2),s,t);
```

$$\mathrm{BesselJ}(0, \sqrt{k^2}\ t)$$

```
> readlib(hypergeom):
> hypergeom([1,2],[2,3],z);
```

$$\mathrm{hypergeom}([\,1\,], [\,3\,], z)$$

```
> simplify(hypergeom([1,2],[2,3],z));
```

$$-\frac{2\ \mathrm{e}^z\ (-1+\mathrm{e}^{(-z)}+\mathrm{e}^{(-z)}\ z)}{z^2}$$

```
> Beta(0.3,0.5);
```

$$4.554443088$$

```
> Ei(1,0.1);
```

$$1.822923958$$

```
> laplace(Dirac(t-3),t,s);
```

$$\mathrm{e}^{(-3\ s)}$$

```
> laplace(Heaviside(t-Pi)*cos(t),t,s);
```

$$-\frac{\mathrm{e}^{(-s\ \pi)}\ s}{s^2+1}$$

1.2.3 Variables

Maple treats a mathematical variable or an unassigned variable as an algebraic unknown. It has no value other than its name. An unassigned variable may be used for such things as a dummy variable, an index in integrals or sums, or as a formal parameter in function definitions. A variable can be assigned a value by using the assignment operator :=. For example, an integer can be assigned to a variable, f, with the command f:=256;. Once a value is assigned to a variable, *Maple* will substitute this value as it executes all subsequent commands in expressions containing that variable. A variable, in this case f, can be unassigned by the command f: =(f). The variable assignment will be cleared only upon the termination of the *Maple* session, or the restart command is invoked. The variable may also be reassigned or specifically unassigned.

The **restart** command permits the start of entirely new *Maple* session and clears all previously assigned variables. This feature is extremely useful when a large number of variables require unassignment but, because the restart command may render all work done in the previous session inaccessible, it must be used with caution.

1.2.4 Numbers, Lists, and Sets

Maple is capable of dealing with integers, and rational, irrational, floating point and complex numbers. Except for floating point numbers, *Maple* performs arithmetic calculations exactly (with no round off error). For floating point arithmetic, the number of decimal digits can be controlled (specified) by assigning the number of digits via the global variable **Digits** where the default value for Digits is 10. *Maple* also allows the user to specify the number of digits in an individual calculation by invoking the command.

A *Maple* list is created by enclosing a sequence in square brackets [ ]. In a set, the sequence is enclosed by braces. There are two basic differences between list and set structures. The first is that the order of elements in a list is preserved whereas the order of elements in a set is arbitrary. For example, when *Maple* returns multiple solutions as a set, the order is determined internally and can vary from one instance to the next even in the same problem. The other difference between a list and a set is that a list may contain duplicated elements, while in a set, duplicate elements are automatically eliminated. An individual element of a list or set can be recalled by using the selection operator [ ]. Example 1.3 illustrates the creation of a list, set and recalling individual elements of both.

Example 1.3

```
> list1:=[sin(x),sin(x),sin(2*x),sin(3*x)];
```

$$list1 := [\sin(x), \sin(x), \sin(2\,x), \sin(3\,x)]$$

```
> set1:={sin(x),sin(x),sin(2*x),sin(3*x)};
```

$$set1 := \{\sin(x), \sin(2\,x), \sin(3\,x)\}$$

```
> list1[4];
```

$$\sin(3\,x)$$

```
> set1[2];
```

$$\sin(2\,x)$$

1.2.5 Algebraic and Transcendental Equations

Maple has the ability to solve a system of algebraic equations, both linear, and within limits, non-linear. The command to be used is **solve**. The solve command involves either one or two arguments. In either case, the first argument is always the equation or the set of equations to be solved. The second command, if used, specifies the variable or set of variables sought. If the second argument is not specified, *Maple* solves for all variables. If *Maple* finds multiple solutions, it returns them as a set and the user must then choose the physically valid solution by using the selection operator, [ ].

Maple will always find a solution to a linear system of equations if the solution exists. Sometimes, however, when dealing with a non-linear system, *Maple* may not return a solution, even when a solution exists. It is important to remember that the solve command does not assign any values to the variables for which it solves and the assign command must be used to assign values to the variables.

Sometimes, *Maple* returns the solution of an algebraic equation in terms of the **RootOf** expression, which is a place holder for all roots of an equation in one variable. The **allvalues** command forces *Maple* to display the roots explicitly. The command **fsolve** can be used to find floating point solutions. Because the solve command cannot obtain the exact roots of a polynomial equation of degree higher than four, or solve transcendental equations, **fsolve** must be used for these situations.

Example 1.4
This example illustrates the solution of algebraic and transcendental equations. When *Maple* is commanded to solve the standard quadratic equation, it gives two roots in terms of a, b, and c. Next, a cubic equation with known coefficients is solved and *Maple* gives an exact solution (sol1) for one real root and two complex conjugate roots. The first root is obtained by using the selection operator, [ ]. When the same equation is solved using the **fsolve** command, *Maple* finds the only real root in the floating point arithmetic. However, if one applies the **evalf** command to sol1, floating point solutions for all three roots appear. Next, we use the **fsolve** command to solve the transcendental equation, sinh(2x) – 6x=0, which arises in the optimum analysis for a straight convecting fin [4]. Lastly, a quartic equation is solved. In this case, *Maple* returns the solution in terms of **RootsOf** expression. The **allvalues** command is combined with the **evalf** command to force *Maple* to display all four complex roots explicitly in floating point arithmetic.

```
> solve(a*x^2+b*x+c=0,x);
```

$$\frac{-b+\sqrt{b^2-4\,a\,c}}{2\,a},\ \frac{-b-\sqrt{b^2-4\,a\,c}}{2\,a}$$

```
> sol1:=solve(2*x^3+3*x^2+4*x+5=0,x);
```

$$sol1 := -\frac{(189+6\sqrt{1086})^{(1/3)}}{6} + \frac{5}{2\,(189+6\sqrt{1086})^{(1/3)}} - \frac{1}{2}, \frac{(189+6\sqrt{1086})^{(1/3)}}{12} - \frac{5}{4\,(189+6\sqrt{1086})^{(1/3)}} - \frac{1}{2} + \frac{1}{2}I\sqrt{3}\left(-\frac{(189+6\sqrt{1086})^{(1/3)}}{6} - \frac{5}{2\,(189+6\sqrt{1086})^{(1/3)}}\right), \frac{(189+6\sqrt{1086})^{(1/3)}}{12} - \frac{5}{4\,(189+6\sqrt{1086})^{(1/3)}} - \frac{1}{2} - \frac{1}{2}I\sqrt{3}\left(-\frac{(189+6\sqrt{1086})^{(1/3)}}{6} - \frac{5}{2\,(189+6\sqrt{1086})^{(1/3)}}\right)$$

```
> sol1[1];
```

$$-\frac{(189+6\sqrt{1086})^{(1/3)}}{6} + \frac{5}{2\,(189+6\sqrt{1086})^{(1/3)}} - \frac{1}{2}$$

```
> fsolve(2*x^3+3*x^2+4*x+5=0,x);
```

$$-1.371134331$$

```
> evalf(sol1);
```

$$-1.371134331, -0.0644328344 - 1.348761012\,I, -0.0644328344 + 1.348761012\,I$$

```
> fsolve(sinh(2*x)-6*x=0,x=1);
```

$$1.419223190$$

```
> sol2:=solve(x^4-3*x^3+9*x^2+19*x+37=0,x);
```

$$sol2 := \text{RootOf}(\_Z^4 - 3\,\_Z^3 + 9\,\_Z^2 + 19\,\_Z + 37, index = 1),$$
$$\text{RootOf}(\_Z^4 - 3\,\_Z^3 + 9\,\_Z^2 + 19\,\_Z + 37, index = 2),$$
$$\text{RootOf}(\_Z^4 - 3\,\_Z^3 + 9\,\_Z^2 + 19\,\_Z + 37, index = 3),$$
$$\text{RootOf}(\_Z^4 - 3\,\_Z^3 + 9\,\_Z^2 + 19\,\_Z + 37, index = 4)$$

```
> evalf(sol2);
```

$$2.449111683 + 3.159874681\, I,\ -0.9491116833 + 1.189181662\, I,\ -0.9491116833 - 1.189181662\, I,\ 2.449111683 - 3.159874681\, I$$

1.2.6 Differential Equations

Maple offers a variety of tools for solving ordinary differential equations through the command **dsolve**. These include the Laplace transformation, the power series method and the Runge-Kutta methods [1]. The first step in solving a differential equation is to enter it in *Maple*. The command used for this purpose is **diff** and the first argument of this command is the function to be differentiated. The second argument is the variable with respect to which the differentiation is to be performed. For second and higher derivatives, a sequence of variables may be used as the remaining arguments.

The **dsolve** command has several forms:

dsolve(equation, depvar(indvar)),
dsolve(set of equations and initial conditions, set of variables),
dsolve(set of equations and initial conditions, set of variables, option)

As mentioned earlier, the set must be enclosed in braces and the option may be **laplace, series, explicit** or **numeric.**

Besides the initial value problems, *Maple* is also capable of solving some boundary value problems. In this case, the boundary conditions replace the initial conditions in the **dsolve** command. However, most non-linear boundary value problems require the *numeric* option and the missing initial condition or conditions must be guessed. In this case, the numerical solution is returned as a *Maple* procedure requiring one argument. When invoked with a numerical value of the argument (the independent variable), the function returns a sequence of numbers giving the values of the independent variable, the dependent variable, and its derivatives. The user must check that the solution satisfies the other known boundary conditions and, if it doesn't, new values for the missing initial conditions must be assumed and the procedure repeated until the solution satisfies the other known boundary condition or conditions. The standard shooting technique for the solution of two point boundary value problems has been recently coded in *Maple*[5].

Example 1.5
This example shows how differential equations can be entered in *Maple*. Eq1 is generated using the command **diff**. Next, the **dsolve** command is invoked to obtain the solution of the differential equation. Because the initial condition was not specified, the solution contains a constant, _C1. In the next statement, the initial condition y(0) = 5 is specified whereupon *Maple* calculates the constant and delivers the solution. Following that, we generate a system of two first-order

differential equations with specified initial conditions and solve them using the numeric option. In this case, *Maple* delivers the solution as a procedure (see section 1.2.8). They symbol rkf45 indicates that *Maple* used a Fehlberg fourth-fifth order Runge-Kutta method [6,7]. The solution for t = 0.1,0.2, and 1.0 are shown. Next we solve the same system of equations using the series option. The last part of the example shows the solution of a boundary value problem. When the boundary conditions are not specified, the solution delivered contains two constants, _C1 and _C2. These constants are automatically evaluated by *Maple* when the boundary conditions are specified.

```
> restart;
> Eq1:=diff(y(t),t)+20*y(t)-7*exp(-1/2*t)=0;
```

$$eq1 := \left(\frac{d}{dt}\,\mathrm{y}(t)\right) + 20\,\mathrm{y}(t) - 7\,\mathrm{e}^{\left(-\frac{t}{2}\right)} = 0$$

```
> dsolve(Eq1,y(t));
```

$$\mathrm{y}(t) = \frac{14}{39}\,\mathrm{e}^{\left(-\frac{t}{2}\right)} + \mathrm{e}^{(-20\,t)}\,\_C1$$

```
> dsolve( {Eq1, y(0)=5}, y(t));
```

$$\mathrm{y}(t) = \frac{14}{39}\,\mathrm{e}^{\left(-\frac{t}{2}\right)} + \frac{181}{39}\,\mathrm{e}^{(-20\,t)}$$

```
> sys:=diff(x(t),t)=x(t)*y(t)+t,diff(y(t),t)=x(t)-t;
```

$$sys := \frac{d}{dt}\,\mathrm{x}(t) = \mathrm{x}(t)\,\mathrm{y}(t) + t,\ \frac{d}{dt}\,\mathrm{y}(t) = \mathrm{x}(t) - t$$

```
> funcs:={x(t),y(t)};
```

$$funcs := \{\mathrm{y}(t), \mathrm{x}(t)\}$$

```
> sol:=dsolve({sys,x(0)=0,y(0)=1},funcs,numeric);
```

$$sol := \mathbf{proc}\ (x\_rkf45)\ \ldots\ \mathbf{end\ proc}$$

```
> sol(0.1);
```

$$[t = 0.1, \mathrm{x}(t) = 0.00517039559380227770, \mathrm{y}(t) = 0.995170919331011094]$$

```
> sol(0.2);
```

$$[t = 0.2, \mathrm{x}(t) = 0.0213862560889388244, \mathrm{y}(t) = 0.981402215649849086]$$

```
> sol(1.0);
```

$$[t = 1.0, x(t) = 0.667366342191922702, y(t) = 0.709633796111630798]$$

```
> dsolve({sys,x(0)=0,y(0)=1},funcs,series);
```

$$\{x(t) = \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 - \frac{1}{24}t^5 + O(t^6), y(t) = 1 - \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \frac{1}{120}t^5 + O(t^6)\}$$

```
> Eq2:=diff(y(x),x,x)-2*diff(y(x),x)+2*y(x)=0;
```

$$Eq2 := \left(\frac{d^2}{dx^2} y(x)\right) - 2\left(\frac{d}{dx} y(x)\right) + 2\, y(x) = 0$$

```
> dsolve(Eq2,y(x));
```

$$y(x) = \_C1\, e^x \sin(x) + \_C2\, e^x \cos(x)$$

```
> dsolve({Eq2,y(0)=-3,y(Pi/2)=0},y(x));
```

$$y(x) = -3\, e^x \cos(x)$$

1.2.7 Manipulation and Simplification

Maple offers a number of commands for the manipulation of polynomials and expressions containing polynomials. Examples of these commands are:

factor, expand, normal, collect, sort, coeff, combine, convert, simplify, and **limit.**

Table 1.2 on page 16 describes the action or actions resulting from these commands.

Example 1.6
This example illustrates the use of commands for manipulating polynomials and expressions containing polynomials.

```
> factor(x^2-2*x-15);
```

$$(x+3)(x-5)$$

```
> expand((x+3)*(x-5));
```

$$x^2 - 2x - 15$$

```
> normal((a^2-b^2)/(a^3-b^3));
```

$$\frac{a+b}{a^2 + a\, b + b^2}$$

```
> p:=expand((2+a-x)*(x+b)^3);
```

$$p := 2\,x^3 + 6\,x^2\,b + 6\,x\,b^2 + 2\,b^3 + a\,x^3 + 3\,a\,x^2\,b + 3\,a\,x\,b^2 + a\,b^3 - x^4 - 3\,x^3\,b - 3\,x^2\,b^2 - x\,b^3$$

```
> collect(p,x);
```

$$-x^4 + (2 + a - 3\,b)\,x^3 + (6\,b + 3\,a\,b - 3\,b^2)\,x^2 + (6\,b^2 + 3\,a\,b^2 - b^3)\,x + 2\,b^3 + a\,b^3$$

```
> sort(p,x);
```

$$-x^4 - 3\,b\,x^3 + 2\,x^3 + a\,x^3 - 3\,b^2\,x^2 + 6\,b\,x^2 + 3\,a\,b\,x^2 - b^3\,x + 6\,b^2\,x + 3\,a\,b^2\,x + 2\,b^3 + a\,b^3$$

```
> coeff(p,x^2);
```

$$6\,b + 3\,a\,b - 3\,b^2$$

```
> expr1:=exp(x)*exp(y)*exp(z);
```

$$expr1 := e^x\,e^y\,e^z$$

```
> combine(expr1,exp);
```

$$e^{(x+y+z)}$$

```
> expr2:=(exp(x)+exp(-x));
```

$$expr2 := e^x + e^{(-x)}$$

```
> convert(%,trig);
```

$$2\cosh(x)$$

```
> simplify(cosh(a)^2-sinh(a)^2);
```

$$1$$

```
> limit(exp(x^2)*erfc(x),x=infinity);
```

$$0$$

1.2.8 Functional Relationships and Procedures

The arrow notation, – >, allows the user to define functions. The statement for defining a function of one variable would be

```
function name:= variable -> expression
```

A function of several variables can be defined by an expression of the form

```
function name:= (var1, var2, ... varn) -> expression.
```

The variables in the function definition are local "dummy" variables and are unrelated to any other variables used elsewhere in the code including those with the same name. Sometimes, the user may wish to convert an expression generated in a *Maple* session into a function. The easiest way to achieve this conversion is through the use of the **unapply** command.

A more general way of defining functions can include a sequence of actions such as the assignment commands **and, if, while, do, for**.

Each statement is a *Maple* procedure *proc*. The general form of a *Maple proc* definition is

```
proc(parameter sequence)
  local variable sequence;
  options option sequence
  statement-1;
  statement-2;
  statement-n;
  end;
```

The parameter sequence can be empty if the procedure does not require any parameters (arguments). The local variable sequence should include all variables used within the procedure that are not formal parameters (arguments). This ensures that they will not be considered as global variables by *Maple.* However, the local sequence is optional. Options available include:
remember, builtin, operator, trace, arrow, angle, and system

The option line can be omitted if not needed.

Example 1.7
This example illustrates the use of the arrow notation and the **unapply** command and the creation of a *Maple* procedure for solving algebraic or transcendental equations using the bisection method [7]. The number of iterations *n* depend on the desired accuracy and the interval of search, *b-a* Note that the use of the construct **if-then-else-fi**, and the termination of **do** loop with **od**. Once

the procedure is constructed, we use it to search for the root of the transcendental equation, sinh(2x) - 6x =0, in the interval 1 to 2.

```
> f:=(x,y,z)->x^2+y^2-3*z^3;
```

$$f := (x, y, z) \rightarrow x^2 + y^2 - 3\,z^3$$

```
> x^2+y^2-3*z^3;
```

$$x^2 + y^2 - 3\,z^3$$

```
> g:=unapply(%,x,y,z);
```

$$g := (x, y, z) \rightarrow x^2 + y^2 - 3\,z^3$$

```
> bisect:=proc(f,a,b,acc)
> left:=a;right:=b;
> n:=evalf(ln((b-a)/acc)/ln(2));
> mid:=(a+b)/2;
> for j from 1 to trunc(n+1) do if
> evalf(subs(x=mid,f(x))*subs(x=left,f(x)))>0 then
left:=mid; else
> right:=mid;fi;mid:=(left+right)/2;od;end:
> f:=x->sinh(2*x)-6*x;
```

$$f := x \rightarrow \sinh(2\,x) - 6\,x$$

```
> bisect(f,1,2,0.0001);
```

$$\frac{46505}{32768}$$

```
> evalf(%);
```

$$1.419219971$$

Table 1.2 Manipulation and Simplification Commands

Command	Action
factor	factors polynomials and expressions containing polynomials.
expand	(i) distributes multiplication and division over additions and subtractions. (ii) applies multiple angle rules to trigonometric and hyperbolic functions. (iii) applies sum of argument rules to exponentials puts
normal	an expression in a numerator over denominator form and divides out the greatest common divisor of numerator and denominator.
collect	groups coefficients of like powers in a polynomial. This command can be used in many ways.
sort	forces *Maple* to write the terms of a polynomial in a specified order such as in ascending powers of variables.
coeff	extracts the coefficient of individual powers of the variables.
combine	combines terms into a single term. With the use of a second argument such as *exp*, *ln*, and *power*, the command serves to combine exponential logarithmic and power series terms respectively.
convert	converts and expression to a different form. For example, convert/trig converts all exponentials to trigonometric and hyperbolic functions.
simplify	simplifies an expression. With a second argument, a specific type of simplification is applied.
limit	gives the limiting value to an expression.

1.3 THE MAPLE LIBRARY

The *Maple* library consists of the standard library, the miscellaneous library, add-on packages, and the share library. The procedures built into the standard library are automatically loaded depending on the needs of the command. For example, when the command **solve** is invoked, *Maple* automatically reads the standard library file **solve.m**. More files may be read automatically as the *solve* command is executed. The miscellaneous library contains less frequently used procedures and must be called using the command **readlib**. For example, **readlib(hypergeom)** had to be executed in Example 1.2 to obtain the hypergeometric function.

Maple offers several special add-on packages such as the linear algebra package, the plots package, and the simplex optimization package. A call to the specific package loads the package into the run-time library. For example, the command **with(intrans)** allows the use of Laplace transforms in Example 1.2. The share library consists of user developed procedures and programs. These can be accessed with the command **share**.

1.4 GRAPHICS

The graphics component of *Maple* is both extensive and powerful. The two-dimensional graphics includes

1. plots of functions of single real variables,
2. plots defined by parametric equations,
3. plots of implicit functions,
4. data plots and
5. animation of two-dimensional graphics objects.

The basic command for a two-dimensional graphics display is **plot** and the procedures associated with this command are loaded automatically. The plot command may also be employed to plot more than one function at the same time. For example, the command

plot(f1(x), f2(x), f3(x), x = 0..5)

when terminated by a semi-colon prompts *Maple* to plot f_1(x), f_2(x), and f_3(x) over the interval, (0,5). The graphs for f_1(x), f_2(x), and f_3(x) are displayed in different colors but the user must self determine which color represents which function. However, it is possible to exercise greater control by using multiple plot results as arguments to the display command.

Example 1.8
In this example the use of the **plot** command to generate a graph of the Legendre polynomials of order 1 through 5 in the interval x=-1 to 1.

```
> with(orthopoly):
> P(1,x);
```

$$x$$

```
> P(2,x);
```

$$-\frac{1}{2}+\frac{3x^2}{2}$$

```
> P(3,x);
```

$$\frac{5}{2}x^3-\frac{3}{2}x$$

```
> P(4,x);
```

$$\frac{3}{8}+\frac{35}{8}x^4-\frac{15}{4}x^2$$

```
> P(5,x);
```

$$\frac{63}{8}x^5-\frac{35}{4}x^3+\frac{15}{8}x$$

```
> plot([P(1,x),P(2,x),P(3,x),P(4,x),P(5,x)],x=-
1..1,color=[black],linestyle=[DOT,DASH,SOLID,DASHDOT,SOLID],
legend=["P(1,x)","P(2,x)","P(3,x)","P(4,x)","P(5,x)"],
thickness=[0,0,0,0,2]);
```

Figure 1.1 shows the output graph.

The command for generating three-dimensional graphics is **plot3d**. This command allows the user to

- generate surfaces defined by functions of two real variables,
- generate curves tubes and surfaces defined by parametric equations,
- generate implicit surfaces defines by an equation,
- generate surfaces from a list of data points and
- show an animation of three-dimensional graphics objects.

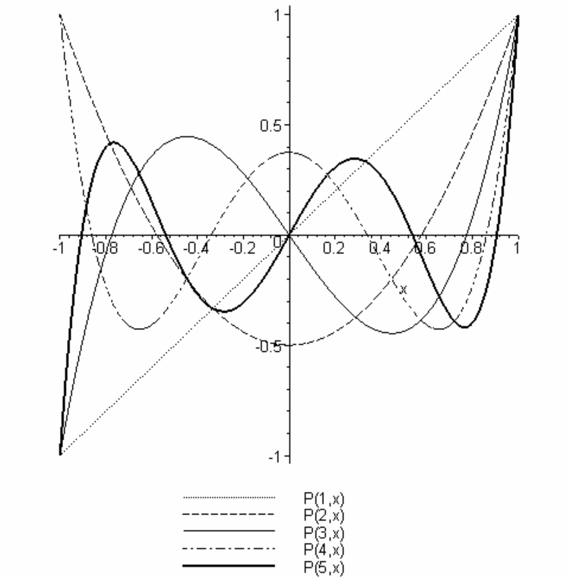

Figure 1.1: Legendre polynomials of order 1 through 5

Example 1.9
This example shows a simple three-dimensional plot. Figure 1.2 shows the output graph.

```
> plot3d(sin(x+y),x=-1..1,y=-1..1);
```

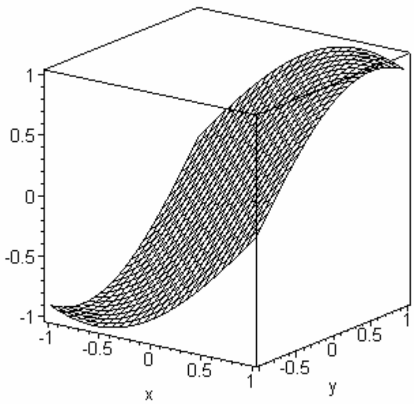

Figure 1.2: Three Dimensional Surface Plot

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Chapter 2

Mathematical Functions in Heat Conduction

2.1 INTRODUCTION

A list of mathematical functions commonly encountered in heat conduction theory was provided in Table 1.1 in Chapter 1. Some basic computations involving these functions were demonstrated in Example 1.2. This chapter deals with more extensive manipulations of these functions. The functions considered are:

- The hyperbolic functions
- The error and complementary error functions
- The GAMMA function
- The Bessel functions
- The generalized hypergeometric function
- The beta function
- The exponential integral function

2.2 HYBERBOLIC FUNCTIONS

Hyperbolic functions arise in conduction-convection systems such as in extended surfaces. Kern and Kraus [1] provide a summary of the relationships between hyperbolic functions that are useful in extended surface heat transfer analysis. Those relationships are the basis for Example 2.1.

Example 2.1

```
> convert(sinh(x),exp,x);
```

$$\frac{1}{2}\mathrm{e}^{x}-\frac{1}{2}\frac{1}{\mathrm{e}^{x}}$$

```
> f:=[sinh(x),cosh(x),tanh(x)];
```

$$f := [\sinh(x), \cosh(x), \tanh(x)]$$

```
> convert(f,exp,x);
```

$$\left[\frac{1}{2}\mathrm{e}^{x}-\frac{1}{2}\frac{1}{\mathrm{e}^{x}}, \frac{1}{2}\mathrm{e}^{x}+\frac{1}{2}\frac{1}{\mathrm{e}^{x}}, \frac{(\mathrm{e}^{x})^{2}-1}{(\mathrm{e}^{x})^{2}+1}\right]$$

```
> h:=[coth(x),sech(x),csch(x)];
```

$$h := [\coth(x), \operatorname{sech}(x), \operatorname{csch}(x)]$$

```
> convert(h,exp,x);
```

$$\left[\frac{(\mathrm{e}^{x})^{2}+1}{(\mathrm{e}^{x})^{2}-1}, \frac{2}{\mathrm{e}^{x}+\frac{1}{\mathrm{e}^{x}}}, \frac{2}{\mathrm{e}^{x}-\frac{1}{\mathrm{e}^{x}}}\right]$$

```
> cosh(-x),sinh(-x),tanh(-x),coth(-x),sech(-x),csch(-x);
```

$$\cosh(x), -\sinh(x), -\tanh(x), -\coth(x), \operatorname{sech}(x), -\operatorname{csch}(x)$$

```
> expr1:=(cosh(x))**2 - (sinh(x))**2;
```

$$expr1 := \cosh(x)^{2}-\sinh(x)^{2}$$

```
> simplify(expr1);
```

$$1$$

```
> expr2:=cosh(x)**2 + sinh(x)**2;
```

$$expr2 := \cosh(x)^{2}+\sinh(x)^{2}$$

```
> simplify(expr2);
```

$$2\cosh(x)^{2}-1$$

```
> cosh(x+y):=expand(cosh(x+y));
```

$$\cosh(x+y) := \cosh(x)\cosh(y) + \sinh(x)\sinh(y)$$

```
> cosh(x-y):=expand(cosh(x-y));
```

$$\cosh(x-y) := \cosh(x)\cosh(y) - \sinh(x)\sinh(y)$$

```
> sinh(x+y):=expand(sinh(x+y));
```

$$\sinh(x+y) := \sinh(x)\cosh(y) + \cosh(x)\sinh(y)$$

```
> sinh(x-y):=expand(sinh(x-y));
```

$$\sinh(x-y) := \sinh(x)\cosh(y) - \cosh(x)\sinh(y)$$

```
> sinh(2*x):=expand(sinh(2*x));
```

$$\sinh(2\,x) := 2\sinh(x)\cosh(x)$$

```
> cosh(2*x):=expand(cosh(2*x));
```

$$\cosh(2\,x) := 2\cosh(x)^2 - 1$$

```
> tanh(2*x):=expand(tanh(2*x));
```

$$\tanh(2\,x) := \frac{2\sinh(x)\cosh(x)}{2\cosh(x)^2 - 1}$$

```
> coth(2*x):=expand(cot(2*x));
```

$$\coth(2\,x) := \frac{1}{2}\,\frac{-1+\cot(x)^2}{\cot(x)}$$

```
> diff(sinh(x),x);
```

$$\cosh(x)$$

```
> diff(cosh(x),x);
```

$$\sinh(x)$$

```
> diff(tanh(x),x);
```

$$1 - \tanh(x)^2$$

```
> diff(coth(x),x);
```

$$1 - \coth(x)^2$$

```
> diff(sech(x),x);
```

$$-\operatorname{sech}(x)\tanh(x)$$

```
> diff(csch(x),x);
```

$$-\operatorname{csch}(x)\coth(x)$$

2.3 THE ERROR FUNCTION

The error function, **erf**, the complementary error function, **erfc**, and the repeated integrals of the complementary error function arise in the analysis of transient heat conduction in a semi-infinite medium [2,3]. These functions also arise when the Laplace transformation is used to analyze transient conduction in a finite slab or in a solid cylinder [2,3]. The derivatives of these functions are needed when evaluating the transient surface heat flux.

The error function for an argument, x, is defined by

$$erf(x) = \frac{2}{\sqrt{\pi}} \cdot \int_0^x e^{-t^2} dt \tag{2.1}$$

where t is a dummy variable. The error function is an odd function. Consequently

$$erf(-x) = -erf(x) \tag{2.2}$$

and

$$erf(0) = 0 \text{ and } erf(\infty) = 1 \tag{2.3}$$

The complimentary error function is defined by

$$erfc(x) = 1 - erf(x) = \frac{2}{\sqrt{\pi}} \cdot \int_x^\infty e^{-t^2} dt \tag{2.4}$$

The derivatives of the error function can be obtained by a repeated differentiation of Eq. (2.1). Thus

$$\frac{d}{dx} erf(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \quad \text{and} \quad \frac{d^2}{dx^2} erf(x) = -\frac{4}{\sqrt{\pi}} x e^{-x^2} \tag{2.5}$$

The repeated integrals of the complimentary error function are defined by

$$i^n erfc(x) = \frac{1}{\sqrt{\pi}} \int_x^\infty i^{n-1} erf(t) dt \quad \text{n = 1,2,3,\dots} \tag{2.6}$$

with

$$i^0 erfc(x) = erfc(x) \quad \text{and} \quad i^{-1} erc(x) = \frac{2}{\sqrt{\pi}} e^{-x^2} \qquad (2.7)$$

For example, the first two repeated integrals are

$$ierfc(x) = \frac{1}{\sqrt{\pi}} e^{-x^2} - xerfc(x) \qquad (2.8)$$

$$i^2 erfc(x) = \frac{1}{4}\left[\left(1 + 2x^2\right) erfc(x) - \frac{2}{\sqrt{\pi}} xe^{-x^2}\right] \qquad (2.9)$$

In Example 2.2, *Maple* is used to derive expressions for the first three derivatives of the error functions. The results are then used to generate a table of numerical values of the derivatives for the interval from x = 0 to x = 1 in increments of 0.1. Example 2.3 shows how a table of the complementary error function and its first three repeated integrals can be generated using *Maple*. Miscellaneous manipulations with the error function are demonstrated in Example 2.4.

Example 2.2

```
> restart; f(x):=erf(x);
```

$$f(x) := erf(x)$$

```
> f[1]:=diff(f(x),x);
```

$$f_1 := \frac{2\, e^{(-x^2)}}{\sqrt{\pi}}$$

```
> F[1]:=unapply(%,x);
```

$$F_1 := x \to \frac{2\, e^{(-x^2)}}{\sqrt{\pi}}$$

```
> f[2]:=diff(f[1],x);
```

$$f_2 := -\frac{4\, x\, e^{(-x^2)}}{\sqrt{\pi}}$$

```
> F[2]:=unapply(%,x);
```

$$F_2 := x \to -\frac{4\,x\,e^{(-x^2)}}{\sqrt{\pi}}$$

```
> f[3]:=diff(f[2],x);
```

$$f_3 := -\frac{4\,e^{(-x^2)}}{\sqrt{\pi}} + \frac{8\,x^2\,e^{(-x^2)}}{\sqrt{\pi}}$$

```
> F[3]:=unapply(%,x);
```

$$F_3 := x \to -\frac{4\,e^{(-x^2)}}{\sqrt{\pi}} + \frac{8\,x^2\,e^{(-x^2)}}{\sqrt{\pi}}$$

```
> for k from 0.1 by 0.1 to 1.0 do
> print(k,erf(k),evalf(subs(x=k,f[1])),evalf(subs(x=k,
f[2])), evalf(subs(x=k,f[3])));od;
```

0.1, 0.1124629160, 1.117151607, -0.2234303213, -2.189617149
0.2, 0.2227025892, 1.084134787, -0.4336539148, -1.994808008
0.3, 0.3286267595, 1.031260910, -0.6187565458, -1.691267892
0.4, 0.4283923550, 0.9615412988, -0.7692330390, -1.307696166
0.5, 0.5204998778, 0.8787825788, -0.8787825788, -0.8787825788
0.6, 0.6038560908, 0.7872434316, -0.9446921179, -0.4408563217
0.7, 0.6778011938, 0.6912748604, -0.9677848046, -0.02765099442
0.8, 0.7421009647, 0.5949857862, -0.9519772579, 0.3331920403
0.9, 0.7969082124, 0.5019685742, -0.9035434336, 0.6224410320
1.0, 0.8427007929, 0.4151074974, -0.8302149948, 0.8302149948

Example 2.3

```
> for k from 0 by 0.1 to 1.0 do
> print(k,erfc(0,k),erfc(1,k),erfc(2,k),erfc(3,k));od;
```

$$0,\ 1,\ \frac{1}{\sqrt{\pi}},\ \frac{1}{4},\ \frac{1}{6\sqrt{\pi}}$$

$$0.1, 0.8875370840, 0.4698220950, 0.1983931662, 0.07169057696$$
$$0.2, 0.7772974108, 0.3866079114, 0.1556635616, 0.05405708113$$
$$0.3, 0.6713732405, 0.3142184826, 0.1207105377, 0.04029869333$$
$$0.4, 0.5716076450, 0.2521275914, 0.09247639295, 0.02969107951$$
$$0.5, 0.4795001222, 0.1996412284, 0.06996472345, 0.02161275082$$
$$0.6, 0.3961439092, 0.1559353704, 0.05225536618, 0.01553815516$$
$$0.7, 0.3221988062, 0.1200982659, 0.03851530848, 0.01102947234$$
$$0.8, 0.2578990353, 0.09117366489, 0.02800529287, 0.007727532718$$
$$0.9, 0.2030917876, 0.06820167830, 0.02008219166, 0.005342288886$$
$$1.0, 0.1572992070, 0.05025454166, 0.01419753093, 0.003643246632$$

Example 2.4

```
> erfc(1,0.3);
```

$$0.3142184826$$

```
> erfc(6,1.0);
```

$$0.00004006518878$$

```
> erfc(3,1.0-0.3*I);
```

$$0.001521036811 + 0.003568296564\, I$$

```
> erf(x):=taylor(erf(x),x=0,10);
```

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}}x - \frac{2}{3\sqrt{\pi}}x^3 + \frac{1}{5\sqrt{\pi}}x^5 - \frac{1}{21\sqrt{\pi}}x^7 + \frac{1}{108\sqrt{\pi}}x^9 + \mathrm{O}(x^{10})$$

```
> expand(erfc(4,x),x);
```

$$\frac{x^4}{24} - \frac{1}{24}x^4\,\mathrm{erf}(x) - \frac{1}{24}\frac{x^3\,\mathrm{e}^{(-x^2)}}{\sqrt{\pi}} + \frac{x^2}{8} - \frac{1}{8}x^2\,\mathrm{erf}(x) - \frac{5}{48}\frac{x\,\mathrm{e}^{(-x^2)}}{\sqrt{\pi}} + \frac{1}{32} - \frac{1}{32}\mathrm{erf}(x)$$

2.4 BESSEL FUNCTIONS

The Bessel differential equation in various forms arises in a number of heat conduction applications. Examples include multidimensional steady and unsteady heat conduction in cylindrical and spherical geometries and convecting extended surfaces. These differential equations have solutions in terms of Bessel functions. An excellent introduction to Bessel functions and their engineering application is given by Andrews [3].

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0 \tag{2.10}$$

is called Bessel's equation of order n. It possesses a solution [4]

$$y = C_1 J_n(x) + C_2 Y_n(x) \tag{2.11}$$

where $J_n(x)$ is the Bessel function of the first kind of order n and argument x and $Y_n(x)$ is the Bessel function of the second kind of order n and argument x. These functions are defined by the infinite series

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (x/2)^{2m+n}}{m!\Gamma(m+n+1)} \tag{2.12}$$

$$Y_n(x) = \frac{J_n(x)\cos(n\pi) - J_{-n}(x)}{\sin(n\pi)} \qquad n \neq 0,1,2,\ldots \tag{2.13a}$$

or

$$Y_n(x) = \lim_{m \to n} \frac{J_m(x)\cos m\pi - J_{-n}(x)}{\sin m\pi} \qquad n = 0,1,2\ldots \tag{2.13b}$$

where $\Gamma(x)$ is the gamma function.

The differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + n^2)y = 0 \tag{2.14}$$

is called the modified Bessel equation of order n. Its solution appears in terms of the modified Bessel functions, $I_n(x)$ (first kind) and $K_n(x)$ (second kind)

$$y = C_1 I_n(x) + C_2 K_n(x) \tag{2.15}$$

These functions are defined by the series

$$I_n(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+n}}{m!\Gamma(m+n+1)} \tag{2.16}$$

$$K_n(x) = \frac{\pi}{2\sin n\pi}\left[I_{-n}(x) - I_n(x)\right] \qquad n \neq 0,1,2,\ldots \tag{2.17}$$

or

$$K_n(x) = \lim_{m\to n} \frac{\pi}{2\sin n\pi}\left[I_{-m}(x) - I_m(x)\right] \quad n = 0,1,2,\ldots \tag{2.18}$$

A convenient way to determine whether a given differential equation possesses a solution in terms of Bessel functions is to compare with the generalized Bessel equation [1-3].

$$\frac{d^2y}{dx^2} + \left[\frac{1-2m}{x} - 2\alpha\right]\frac{dy}{dx} + \left[p^2a^2x^{2p-2} + \alpha^2 + \frac{\alpha(2m-1)}{x} + \frac{m^2 - p^2n^2}{x^2}\right]y = 0 \tag{2.19}$$

which has a solution of the form

$$y = x^m e^{\alpha x}\left[C_1 J_n ax^p + C_2 Y_n\left(ax^p\right)\right]$$

Maple is capable of solving many forms of Bessel equations in terms of $J_n(x)$,$Y_n(x)$,$I_n(x)$, and $K_n(x)$. The commands for these functions **are BesselJ, BesselY, BesselI**, and **BesselK** respectively. However, the **alias** command can be used to abbreviate these commands to *J, Y, I,* and *K.*

Each Bessel command has two arguments. The first argument is the order of the function while the second argument is the variable. For example, **BesselK**(2,x) represents $K_2(x)$. Example 2.5 shows how equations (2.10), (2.11), (2.14), and (2.15) can be generated using *Maple.* In Example 2.6*, Maple* is employed to develop some relationships involving the derivatives of the Bessel functions. Example 2.7 uses *Maple* to develop some relationships involving integrals of the Bessel functions. Finally, Example 2.8 shows how graphs of $J_o(x)$, $J_1(x)$, and $J_2(x)$ can be generated.

Example 2.5

```
> restart;
```

```
>alias(J=BesselJ,Y=BesselY,K=BesselK,I=BesselI,C[1]=_C1,C[2]
=_C2);
```

$$J, Y, K, I, C_1, C_2$$

```
> Eq1:=x**2*diff(y(x),x,x)+x*diff(y(x),x)+(x**2-
n**2)*y(x)=0;
```

$$Eq1 := x^2\left(\frac{d^2}{dx^2}\,y(x)\right) + x\left(\frac{d}{dx}\,y(x)\right) + (x^2 - n^2)\,y(x) = 0$$

```
> dsolve(Eq1,y(x));
```

$$y(x) = C_1\,J(n,x) + C_2\,Y(n,x)$$

```
> Eq2:=x**2*diff(y(x),x,x)+x*diff(y(x),x)-
(x**2+n**2)*y(x)=0;
```

$$Eq2 := x^2\left(\frac{d^2}{dx^2}\,y(x)\right) + x\left(\frac{d}{dx}\,y(x)\right) - (x^2 + n^2)\,y(x) = 0$$

```
> dsolve(Eq2,y(x));
```

$$y(x) = C_1\,I(n,x) + C_2\,K(n,x)$$

Example 2.6

```
> diff(J(n,x),x);
```

$$-J(n+1,x) + \frac{n\,J(n,x)}{x}$$

```
> simplify(diff(x**n*J(n,x),x));
```

$$\frac{2\,x^n\,n\,J(n,x) - x^{(n+1)}\,J(n+1,x)}{x}$$

```
> diff(K(n,x),x);
```

$$-K(n+1,x) + \frac{n\,K(n,x)}{x}$$

```
> simplify(diff(x**n*K(n,x),x));
```

$$\frac{2\,x^n\,n\,K(n,x) - x^{(n+1)}\,K(n+1,x)}{x}$$

Example 2.7

```
> int(x**(n+1)*J(n,x),x);
```

$$x^{(n+1)}\,J(n+1,x)$$

```
> int(x**(-n)*J(n+1,x),x);
```

$$-x^{(-n)}\,J(n,x)$$

```
> int(x**(n)*Y(n-1,x),x);
```

$$x^n\,Y(n,x)$$

```
> int(x**(-n)*Y(n+1,x),x);
```

$$-x^{(-n)}\,Y(n,x)$$

```
> int(I(1,m*x),x);
```

$$\frac{I\;(0,m\,x)}{m}$$

```
> int(K(1,m*x),x);
```

$$-\frac{K(0,m\,x)}{m}$$

```
> int(x**2*J(1,x),x);
```

$$x^2\,J(2,x)$$

```
> int(J(3,x),x);
```

$$1 - J(0,x) - 2\,J(2,x)$$

Example 2.8 provides a plot of $J_0(x)$, $J_1(x)$, and $J_2(x)$. The plot is shown in Figure 2.1 where $J_0(x)$ is Bessel J(0,x), $J_1(x)$ is Bessel J(1,x), and $J_2(x)$ is Bessel J(2,x). Note the use of the linestyle, color, title, and legend arguments. In order to use the legend command with multiple equations one must use [ ] to enclose a list of equations, not the set of equations enclosure, { }.

Example 2.8:

```
>plot([J(0,x),J(1,x),J(2,x)],x=0..15,title="Bessel Function
Plots",linestyle=[DOT,DASH,SOLID],color=black,

legend=["Bessel J(0,x)","Bessel J(1,x)","Bessel J(2,x)"] );
```

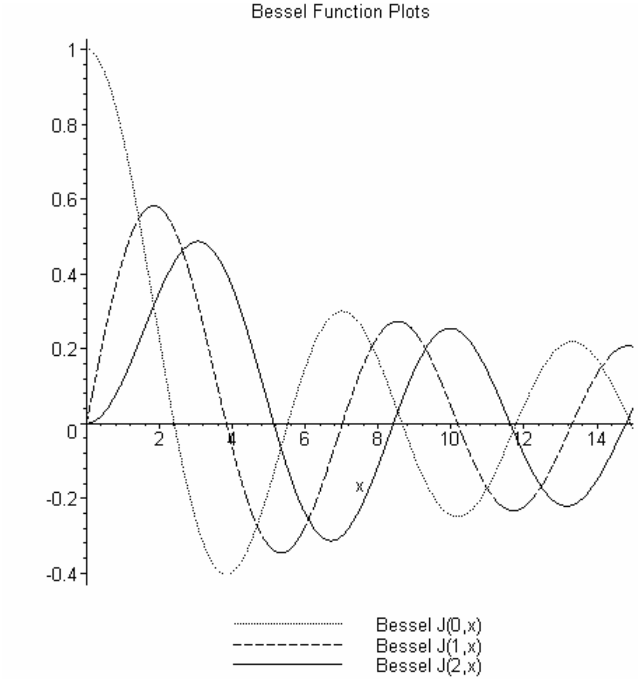

Figure 2. 1 Plot of the Bessel Functions $J_0(x)$, $J_1(x)$, and $J_2(x)$.

2.5 THE EXPONENTIAL INTEGRAL FUNCTION

The exponential integral function appears in the solution of transient conduction in an infinitely long cylindrical body with a uniform line heat source placed at its axis [2, 3]. The exponential integral, $E_1(x)$ or $-Ei(x)$ for a real argument, x is defined as [2-4].

$$E_1(x) = -Ei(x) = \int_x^{\infty} \frac{e^{-t}}{t}\,dt \tag{2.20}$$

with

$$E_1(0) = \infty \text{ and } E_1(\infty) = 0 \tag{2.21}$$

The function increases monotonically from the value of infinite at $x=0$ to zero as x approaches infinity. Example 2.9 illustrates the use of *Maple* for manipulations and calculations involving the exponential integral function. First, $Ei(x)$ is differentiated with respect to x. Next, a series expansion for $Ei(x)$ up to $O(x^5)$ is generated. Then, a sample value for $x= 1.0$ is calculated and finally, a table of $E_1(x)$ for the interval $x=0.01$ to $x=0.1$ is created.

Example 2.9

```
> diff(Ei(x),x);
```

$$\frac{e^x}{x}$$

```
> series(Ei(x),x=0,5);
```

$$(\gamma + \ln(x)) + x + \frac{1}{4}x^2 + \frac{1}{18}x^3 + \frac{1}{96}x^4 + O(x^5)$$

```
> Ei(1,1.);
```

0.2193839344

```
> for k from 0.01 by 0.01 to 0.1 do
> print(k,Ei(1,k));od;
```

0.01, 4.037929577
0.02, 3.354707783
0.03, 2.959118724

0.04, 2.681263689
0.05, 2.467898489
0.06, 2.295306918
0.07, 2.150838180
0.08, 2.026941003
0.09, 1.918744770
0.10, 1.822923958

2.6 THE GAMMA AND THE INCOMPLETE GAMMA FUNCTION

The gamma function is defined by the integral [4]

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt \quad x > 0 \tag{2.22}$$

while the incomplete gamma function is defined by the integral [4]

$$\Gamma(a,x) = \int_x^\infty e^{-t} t^{a-1} dt \quad a > 0 \tag{2.23}$$

These functions arise when the classical method of separation of variables is applied to heat conduction in spherical regions. They also arise in the analysis of fins with power-law type heat dissipation. The *Maple* syntaxes for these functions are **GAMMA**(x) and **GAMMA**(a,x). Example 2.10 demonstrates the use of *Maple* in generating these functions.

Example 2.10

```
> GAMMA(4);
```

$$6$$

```
> GAMMA(n+1);
```

$$\Gamma(n+1)$$

```
> convert(GAMMA(n+1),factorial);
```

$$\frac{(n+1)!}{n+1}$$

```
> GAMMA(1,-2);
```

$$e^2$$

```
> for k from 1 by 0.1 to 2 do print(k,GAMMA(k));od;
```

1, 1
1.1, 0.9513507699
1.2, 0.9181687424
1.3, 0.8974706963
1.4, 0.8872638175
1.5, 0.8862269255
1.6, 0.8935153493
1.7, 0.9086387329
1.8, 0.9313837710
1.9, 0.9617658319
2.0, 1.

```
> GAMMA(1/2);
```

$$\sqrt{\pi}$$

```
> simplify(GAMMA(x)*GAMMA(1-x));
```

$$\frac{\pi}{\sin(\pi x)}$$

```
> simplify(GAMMA(1+x)*GAMMA(1-x));
```

$$-\frac{\pi x}{\sin(\pi(1+x))}$$

2.7 THE BETA FUNCTION

A useful function of two variables, *x* and *y* is the beta function defined as [4]

$$B(x,y)=\int_0^1 (1-t)^{x-1}t^{y-1}dt=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{2.24}$$

where x>0, y>0. Because of its relationship to the gamma function, the use of the beta function is often overshadowed. However, it is important in heat conduction as it appears in the analysis of radiating fins of rectangular profile [7]. The *Maple* syntax for the beta function is **Beta**. Example 2.11 shows some manipulations involving the beta function.

Example 2.11

```
> Beta(x,y):=convert(Beta(x,y),GAMMA);
```

$$\mathrm{B}(x,y) := \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

```
> Beta(0.3,0.5);
```

$$4.554443088$$

```
> series((GAMMA(2,x),x=0,5));
```

$$1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \mathrm{O}(x^5)$$

2.8 THE GENERALIZED HYPERGEOMETRIC FUNCTION

The hypergeometric function, *F(a,b,c,x)*, is defined by the hypergeometric series

$$F(a,b;c,x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \qquad |x| < 1 \tag{2.25}$$

where $(a)_n$, for example, is

$$(a)_n = (a+1)\cdots(a+n-1) \qquad n = 1,2,3,\cdots$$

The semicolons separate the numerator parameters *a* and *b*, the denominator parameter, *c*, and the argument *x*. This function satisfies the following second order linear differential equation

$$x(1-x)\frac{d^2 y}{dt^2} + [c-(a+b+1)x]\frac{dy}{dx} - aby = 0 \tag{2.26}$$

provided that $c = 0,-1,-2,\cdots$. Equation (2.26) arises, for example, in the two-dimensional analysis of a triangular convecting fin [8] and in the analysis of radiating fins [5,6].

In *Maple*, the hypergeometric function is summoned by the command **hypergeom([][],x)** where the parameters in the numerator appear in the first list and the second list contains the parameters in the denominator. For example, the command **hypergeom([a,b],[c],x)** would refer to the function *F(a,b;c,x)*. Several manipulations involving the hypergeometric function are illustrated in Example 2.12. Note that the library, **hypergeom** must be called first.

Example 2.12

```
> readlib(hypergeom):
> hypergeom([1,2],[2,3],x);
```

$$\mathrm{hypergeom}([1],[3],x)$$

```
> simplify(%);
```

$$-\frac{2\,e^{x}\,(-1+e^{(-x)}+e^{(-x)}\,x)}{x^2}$$

```
> hypergeom([1/2,7/10],[3/2],1);
```

$$\mathrm{hypergeom}\left(\left[\frac{1}{2},\frac{7}{10}\right],\left[\frac{3}{2}\right],1\right)$$

```
> simplify(%);
```

$$\frac{1}{2}\,\frac{\pi^{(3/2)}}{\sin\left(\frac{3\,\pi}{10}\right)\Gamma\left(\frac{7}{10}\right)\Gamma\left(\frac{4}{5}\right)}$$

```
> evalf(%);
```

$$2.277221543$$

```
> hypergeom([a,b],[c],1);
```

$$\mathrm{hypergeom}([a,b],[c],1)$$

```
> simplify(%);
```

$$\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}$$

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Chapter 3

Algebraic and Transcendental Equations

3.1 INTRODUCTION

The finite difference method is widely used for the solution of ordinary and partial differential equations of heat conduction when these equations cannot be solved analytically or the analytical method is too cumbersome to use. The method uses the Taylor series expansion or the control volume approach to replace the derivatives by their discrete approximations. This allows the problem in heat conduction governed by a single set of differential equations to be reduced to a system of algebraic equations, linear or nonlinear depending on the process. Thus, knowledge of the techniques of solving a system of algebraic equations is critical if the temperature field is to be accurately predicted [1].

The heat conduction analyst also frequently encounters transcendental equations that must be solved as part of a heat conduction problem. For example, when the method of separation of variables is applied to solve a multi-dimensional steady heat conduction problem or a one-dimensional transient heat conduction problem, the procedure entails the determination of eigenvalues. These eigenvalues are the roots of transcendental equations containing trigonometric functions if the geometry is Cartesian, and Bessel functions if the geometry is cylindrical [2,3]. The same situation applies when convecting fins of different profiles are optimized for maximum heat dissipation for a given volume of material [4]. The analysis of conduction controlled freezing and melting gives rise to transcendental equations containing the error functions and exponential integrals.

The objective of this chapter is to illustrate, with several examples, how *Maple* can be used to solve a system of algebraic equations and transcendental equations.

3.2 SYSTEM OF ALGEBRAIC EQUATIONS

The techniques that *Maple* employs for the solving of a system of algebraic equations are contained in a separate package called **linalg**. The package can be called with the command **with(linalg)**. One then has the choice of using the method of matrix inversion, Gauss elimination or Gauss-Jordan elimination (reduced row echelon form).

The Gauss elimination method can be implemented either manually using *Maple*'s **addrow**, **mulrow**, **swaprow** and **pivot** commands on the augmented matrix or automatically using the command gausselim. The command for applying the Gauss-Jordan procedure is **gaussjord** or its synonym, **rref**, which is an abbreviation for reduced row echelon form. It is also possible to solve the system of algebraic equations by invoking the **solve** command and let *Maple* use the procedure incorporated into its solve command.

Three examples are presented in order to illustrate the different approaches that can be used to solve a system of algebraic equations.

Example 3.1

```
> with(linalg):
```

Warning, the protected names norm and trace have been redefined and unprotected

```
> A:=matrix(4,4,[-1,1,0,0,1,-2,1,0,0,1,-2,1,0,0,1,-2]);
```

$$A := \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}$$

```
> B:=matrix(4,1,[-40,-30,-30,-30]);
```

$$B := \begin{bmatrix} -40 \\ -30 \\ -30 \\ -30 \end{bmatrix}$$

```
> (AB):=augment(A,B);
```

$$AB := \begin{bmatrix} -1 & 1 & 0 & 0 & -40 \\ 1 & -2 & 1 & 0 & -30 \\ 0 & 1 & -2 & 1 & -30 \\ 0 & 0 & 1 & -2 & -30 \end{bmatrix}$$

```
> G:=gausselim((AB));
```

$$G := \begin{bmatrix} -1 & 1 & 0 & 0 & -40 \\ 0 & -1 & 1 & 0 & -70 \\ 0 & 0 & -1 & 1 & -100 \\ 0 & 0 & 0 & -1 & -130 \end{bmatrix}$$

```
> T:=backsub(G);
```

$$T := [340, 300, 230, 130]$$

Example 3.2: Solve the problem in Example 3.1 by the Gauss-Jordan method

```
> gaussjord((AB));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 340 \\ 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 230 \\ 0 & 0 & 0 & 1 & 130 \end{bmatrix}$$

```
> rref((AB));
```

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 340 \\ 0 & 1 & 0 & 0 & 300 \\ 0 & 0 & 1 & 0 & 230 \\ 0 & 0 & 0 & 1 & 130 \end{bmatrix}$$

Example 3.3: Solve the problem in example 3.1 using the solve command

```
> restart;
> eq1:=-T[1]+T[2]=-40;eq2:=T[1]-2*T[2]+T[3]=-30;eq3:=T[2]-
2*T[3]+T[4]=-30;eq4:=T[3]-2*T[4]=-30;
```

$$eq1 := -T_1 + T_2 = -40$$

$$eq2 := T_1 - 2\,T_2 + T_3 = -30$$

$$eq3 := T_2 - 2\,T_3 + T_4 = -30$$

$$eq4 := T_3 - 2\,T_4 = -30$$

```
> solve({eq1,eq2,eq3,eq4});
```

$$\{T_4 = 130, T_2 = 300, T_3 = 230, T_1 = 340\}$$

3.3 TRANSCENDENTAL EQUATIONS

A number of methods are available for solving nonlinear algebraic equations which also include the transcendental equations [5]. The popular methods which are described in most of the texts on numerical methods are the secant method, the bisection method, and Newton's method. These methods can be programmed using *Maple* and then used to solve the equations. Alternatively, the *Maple* command **fsolve** can be applied directly to the equations. In section 3.3.1, the secant, bisection and Newton methods are employed to solve a transcendental equation (Examples 3.4 to 3.6). In Section 3.3.2, a wide variety of transcendental equations that arise in conduction are solved (Examples 3.7 to 3.14).

3.3.1 Numerical Methods

3.3.1.1 The Secant Method

Consider the nonlinear algebraic equation or transcendental equation

$$f(x)=0 \tag{3.1}$$

The general expression for iteration by the secant method can be written as [5]

$$x_{n+1}=x_n-f(x_n)\left[\frac{x_n-x_{n+1}}{f(x_n)-f(x_{n+1})}\right], \quad n=0,1,2,3\cdots \tag{3.2}$$

This method requires the analyst to specify two initial guesses, x_o and x_1. Example 3.4 illustrates the application of the method to solve a transcendental equation that arises when a convecting fin of rectangular profile is optimized for maximum heat dissipation for a given volume of material [4]. The equation is:

$$\sinh(2N)-6N=0 \tag{3.3}$$

Example 3.4:

```
> f:=sinh(N*2)-6*N;
```

$$f:=\sinh(2\,N)-6\,N$$

```
> evalf(subs(N=1.9,f));
```

$$10.93940686$$

```
> N[0]:=2;
```

$$N_0:=2$$

```
> N[1]:=1.9;
```

$$N_1 := 1.9$$

```
> for n from 1 to 10 by 1 do
> N[n+1]:=N[n]-evalf(subs(N=N[n],f)*(N[n]-N[n-1])/(subs(N=
N[n],f)-subs(N=N[n-1],f)));od;
```

$$N_2 := 1.648548883$$

$$N_3 := 1.524841090$$

$$N_4 := 1.448049202$$

$$N_5 := 1.423394003$$

$$N_6 := 1.419401810$$

$$N_7 := 1.419224324$$

$$N_8 := 1.419223190$$

$$N_9 := 1.419223190$$

Error, numeric exception: division by zero

Note that the error message occurs because the last two values of N are equal and Eq. (3.2) involves a division by zero.

3.3.1.2 The Bisection Method

This method also applies to the solution of Eq. (3.1). It searches for a root in a given interval, say between $x=a$ and $x=b$ and it is based upon the fact that, when an interval, [a,b] has a root, the sign of $f(x)$ at the two ends of the interval must change, that is

$$f(a)f(b) \le 0$$

The first step in the solution procedure is to bisect the interval, [a,b], into two halves, namely, [a,m] and [m,b] where

$$m = \frac{a+b}{2}$$

Next, the method computes *f(a)f(m)* and *f(m)f(b)*and checks their signs, If *f(a)f(m)≤0* , the root lies in the interval [a,m]; otherwise the interval [m,b] contains the root. The new interval containing the root is bisected again and the procedure is repeated. The interval size after n iterations becomes

$$\frac{b-a}{2^n} \tag{3.4}$$

where *b-a* is the initial size of the interval of search. For a specified accuracy or tolerance ε the number of iterations is given by

$$n \geq \ln\left(\frac{\frac{b-a}{\varepsilon}}{\ln(2)}\right) \tag{3.5}$$

In Example 3.5, a bisection procedure in *Maple* is developed and then used to solve Eq. (3.3).

Example 3.5:

```
> restart;
> bisec:=proc(f,a,b,epsilon);
> left:=a;right:=b;
> n:=evalf(ln((b-a)/epsilon)/ln(2));
> m:=(a+b)/2;
> for j from 1 to trunc(n+1) do if
evalf(subs(N=m,f(N))*subs(N=left,f(N)))>0 then left:=m;else
right:=m;fi;m:=(left+right)/2;od;end proc;
```

```
bisec := proc (f, a, b, ε)
local left, right, n, m, j;
    left := a;
    right := b;
    n := evalf( ln( (b − a)/ε )/ln( 2 ) );
        m := 1/2×a + 1/2×b;
        for j to trunc( n + 1 ) do
            if 0 < evalf( subs( N = m, f( N ) )×subs( N = left, f( N ) ) ) then left := m
            else right := m
            end if ;
        m := 1/2×left + 1/2×right
    end do
end proc
```

```
> f:=N->sinh(2*N)-6*N;
```

$$f := N \to \sinh(2\,N) - 6\,N$$

```
> bisec(f,1,2,0.00001);
```

$$\frac{372041}{262144}$$

```
> evalf(%);
```

$$1.419223785$$

3.3.1.3 Newton's Method

In Newton's method, which also applies to the solution of Eq. (3.1), the initial guesses for x, namely x_0, is used to evaluate $f(x_0)$, and its derivative $f'(x_o)$. The iteration then proceeds in accordance with

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \qquad n = 0,1,2,3\cdots \tag{3.6}$$

Example 3.6 shows how Newton's method can be programmed in *Maple* and then used to solve Eq. (3.3).

Example 3.6:

```
> f(N):=sinh(2*N)-6*N;
```

$$\mathrm{f}(N) := \sinh(2\,N) - 6\,N$$

```
> fp(N):=diff(f(N),N);
```

$$\mathrm{fp}(N) := 2\cosh(2\,N) - 6$$

```
> N[0]:=1;
```

$$N_0 := 1$$

```
> for n from 0 to 10 by 1 do
> N[n+1]:=N[n]-evalf(subs(N=N[n],f(N)/fp(N)))
> ;od;
```

```
> seq(N(n),n=0..19);
```

$$N_1 := 2.556778410$$
$$N_2 := 2.133816789$$
$$N_3 := 1.783999231$$
$$N_4 := 1.546341060$$
$$N_5 := 1.439552725$$
$$N_6 := 1.419833391$$
$$N_7 := 1.419223758$$
$$N_8 := 1.419223190$$
$$N_9 := 1.419223190$$
$$N_{10} := 1.419223190$$
$$N_{11} := 1.419223190$$

3.3.2 Direct Solutions

The **fsolve** command in *Maple* provides a direct method for the solution of transcendental equations. In this section, the command is used to solve a five transcendental equations arising in heat conduction theory.

3.3.2.1 Transient Conduction in a Plane wall

In the analysis of one-dimensional conduction in a slab, two transcendental equations require solution. Depending upon the boundary conditions on the two faces of the slab, they are:

$$\lambda \tan \lambda - Bi = 0 \tag{3.7}$$

and

$$\lambda + Bi \tan \lambda = 0 \tag{3.8}$$

Example 3.7 demonstrates the use of *Maple* in obtaining the first root of these equations for Bi = 1, 2, 3, 4, 5. Note that for both equations, the interval of search is specified to insure that *Maple* finds the first root and others.

Example 3.7:

```
> Eq1:= lambda*tan(lambda)-Bi;
```

$$Eq1 := \lambda \tan(\lambda) - \text{Bi}$$

```
> Eq2:=lambda+Bi*tan(lambda);
```

$$Eq2 := \lambda + \text{Bi} \tan(\lambda)$$

```
> for Bi from 1 to 5 do
> fsolve(Eq1=0,lambda,0..2);od;
```

0.8603335890
1.076873986
1.192458829
1.264591571
1.313837716

```
> for Bi from 1 to 5 do
> fsolve(Eq2=0,lambda,1..3)od;
```

2.028757838
2.288929728
2.455643863
2.570431560
2.653662400

3.3.2.2 Transient Conduction in a Cylinder

In the analysis of one-dimensional conduction in a cylinder, the three transcendental equations that normally appear are

$$\lambda J_1(\lambda) - BiJ_0(\lambda) = 0 \tag{3.9}$$

$$J_0(\lambda) = 0 \tag{3.10}$$

and

$$J_0(\lambda)Y_0(2\lambda) - Y_0(\lambda)J_0(2\lambda) = 0 \tag{3.11}$$

Equations Eq. (3.9) and Eq. (3.10) are applicable to the case of a solid cylinder while equation Eq. (3.11) is encountered in the analysis of a hollow cylinder when the radius ratio is 2.

In Example 3.8, *Maple* is used to obtain the first root of Eq. (3.9) for Bi = 1, 2, 3, 4, 5. The first six roots are found for Eq. (3.9) and (3.10) with Bi = 1. Finally, for Eq. (3.11), the first five roots are found. For each equation, the interval of search is based upon a prior knowledge of the root location. However, such knowledge is not necessary. For example, the approximate location of the roots of $J_0(\lambda) = 0$ can be found by plotting the function using the **plot** command.

Example 3.8:

```
> restart;
> Eq3:=lambda*BesselJ(1,lambda)-Bi*BesselJ(0,lambda);
```

$$Eq3 := \lambda\,\text{BesselJ}(1, \lambda) - \text{Bi}\,\text{BesselJ}(0, \lambda)$$

```
> for Bi from 1 to 5 do
> fsolve(Eq3=0,lambda,1..3);od;
```

1.255783712
1.599449206
1.788657173
1.908078791
1.989814715

```
> Eq4:=BesselJ(0,lambda);
```

$$Eq4 := \text{BesselJ}(0, \lambda)$$

```
> guesses:=[2..3,5..6,8..9,11..12,14..15,18..19];
```

$$guesses := [2 .. 3, 5 .. 6, 8 .. 9, 11 .. 12, 14 .. 15, 18 .. 19]$$

```
> approx:=proc(guess)
> fsolve(Eq4=0,lambda,guess) end:
> map(approx,guesses);
```

[2.404825558, 5.520078110, 8.653727913, 11.79153444, 14.93091771, 18.07106397]

```
> restart;
> Eq5:=lambda*BesselJ(1,lambda)-BesselJ(0,lambda);
```

$$Eq5 := \lambda\,\text{BesselJ}(1, \lambda) - \text{BesselJ}(0, \lambda)$$

```
> guesses:=[1..2,3..5,6..8,10..12,13..14,16..17];
```

$$guesses := [1 .. 2, 3 .. 5, 6 .. 8, 10 .. 12, 13 .. 14, 16 .. 17]$$

```
> approx:=proc(guess)
> fsolve(Eq5=0,lambda,guess)
```

```
> end:
> map(approx,guesses);
```

$[1.255783712, 4.079477711, 7.155799175, 10.27098536, 13.39839749, 16.53115893]$

```
> restart;
> Eq6:=BesselJ(0,lambda)*BesselY(0,2*lambda)-BesselY(0,
lambda)*BesselJ(0,2*lambda);
```

$$Eq6 := \text{BesselJ}(0, \lambda)\,\text{BesselY}(0, 2\,\lambda) - \text{BesselY}(0, \lambda)\,\text{BesselJ}(0, 2\,\lambda)$$

```
> guesses:=[3..4,6..7,9..10,12..13,15..16];
```

$$guesses := [3\,..\,4, 6\,..\,7, 9\,..\,10, 12\,..\,13, 15\,..\,16]$$

```
> approx:=proc(guess)
> fsolve(Eq6=0,lambda,guess)
> end:
> map(approx,guesses);
```

$$[3.123030920, 6.273435714, 9.418207542, 12.56142319, 15.70399789]$$

3.3.2.3 Optimum Design of Fins

When convecting fins and spines are optimized for maximum heat dissipation for a given volume of material, the evaluation of the fin parameter, N, which establishes the optimum dimensions, requires the solution of transcendental equations involving hyperbolic and Bessel functions. Some of these equations are:

For longitudinal fins of rectangular profile

$$6N(1-H^2)-(1+H^2)\sinh 2N - 2H(1+\cosh 2N) = 0 \tag{3.12}$$

$$6N(1-G)-(1-3G)\sinh 2N = 0 \tag{3.13}$$

and

$$N(\tanh N - 3N\sec h^2 N) - 2Bi_w = 0 \tag{3.14}$$

For the cylindrical spine,

$$\sinh 2N - \frac{5+n}{3-n}(2N) = 0 \tag{3.15}$$

For the longitudinal fin of triangular profile,

$$I_0^2(2N) - \frac{4}{3}\left[\frac{I_0(2N)I_1(2N)}{2N}\right] - I_1^2(2N) = 0 \tag{3.16}$$

and

$$3N^2\left[\frac{I_1(4N}{I_0(4N)}\right]^2 + N\frac{I_1(4N)}{I_0(4N)} - 3N^2 - \frac{1}{4}Bi_w = 0 \tag{3.17}$$

For the longitudinal fin of convex parabolic profile,

$$3N^2\left[\frac{I_{2/3}(4N}{I_{-1/3}(4N)}\right]^2 + N\frac{I_{2/3}(4N)}{I_{-1/3}(4N)} - 3N^2 - \frac{2}{3}Bi_w = 0 \tag{3.18}$$

For the longitudinal fin of concave parabolic profile,

$$-1 + \sqrt{1+36N^2} - \frac{27N^2}{\sqrt{1+36N^2}} - Bi_w = 0 \tag{3.19}$$

Example 3.9 deals with Eq. (3.12) to (3.14) which pertain to the longitudinal fin of rectangular profile. Eq. (3.15 for the cylindrical spine is considered in Example 3.10 and Eq. (3.16) and (3.17) are the subject of Example 3.11. The solutions for Eq. (3.18) and (3.19) for the longitudinal fins of parabolic profile appear in Example 3.12.

Example 3.9

```
> restart;
> Eq1:=6*N*(1-H**2)-(1+H**2)*sinh(2*N)-2*H*(1+cosh(2*N));
```

$$Eq1 := 6\,N\,(1-H^2) - (1+H^2)\sinh(2\,N) - 2\,H\,(1+\cosh(2\,N))$$

```
> for H from 0 by 0.05 to 0.25 do
```

```
> print(H,fsolve(Eq1=0,N=1.1));od;
```

0, 1.419223190
0.05, 1.331094126
0.10, 1.237076797
0.15, 1.134084547
0.20, 1.016185605
0.25, 0.8676350720

```
> Eq2:=6*N*(1-G)-(1-3*G)*sinh(2*N);
```

$$Eq2 := 6\,N\,(1-G)-(1-3\,G)\sinh(2\,N)$$

```
> for G from 0 by 0.1 to 0.3 do
> print(G,fsolve(Eq2=0,N=2.7));od;
```

0, 1.419223190
0.1, 1.605704552
0.2, 1.913812378
0.3, 2.714772000

```
> restart;
> Eq3:=N*(tanh(N)-3*N*sech(N)**2)-2*Bi[w];
```

$$Eq3 := N\,(\tanh(N)-3\,N\,\mathrm{sech}(N)^2)-2\,\mathrm{Bi}_w$$

```
> for Bi[w] from 0 by 0.1 to 0.5 do
> print(Bi[w],fsolve(Eq3=0,N=1..2));od;
```

0, 1.419223190
0.1, 1.534924413
0.2, 1.643691626
0.3, 1.749081269
0.4, 1.853258124
0.5, 1.957730643

Example 3.10

```
> restart;
> Eq4:=sinh(2*N)-(5+n)*2*N/(3-n);
```

$$Eq4 := \sinh(2\,N)-\frac{2\,(5+n)\,N}{3-n}$$

```
> for n from 0 by 0.1 to 0.5 do
> print(n,fsolve(Eq4=0,N=1));od;
```

0, 0.9192963573
0.1, 0.9711690535
0.2, 1.022009233
0.3, 1.072091723
0.4, 1.121654684
0.5, 1.170911055

Example 3.11

```
> restart;
> Eq5:=BesselI(0,2*N)**2-
(4/3)*(BesselI(0,2*N)*BesselI(1,2*N)/(2*N))-
BesselI(1,2*N)**2;
```

$$Eq5 := \mathrm{BesselI}(0, 2\,N)^2 - \frac{2}{3}\,\frac{\mathrm{BesselI}(0, 2\,N)\,\mathrm{BesselI}(1, 2\,N)}{N} - \mathrm{BesselI}(1, 2\,N)^2$$

```
> fsolve(Eq5=0,N=2);
```

1.309402063

```
> restart;
> Eq6:=3*N**2*(BesselI(1,4*N)/BesselI(0,4*N))**2+N*
(BesselI(1,4*N)/BesselI(0,4*N))-3*N**2-1/4*Bi[w];
```

$$Eq6 := \frac{3\,N^2\,\mathrm{BesselI}(1, 4\,N)^2}{\mathrm{BesselI}(0, 4\,N)^2} + \frac{N\,\mathrm{BesselI}(1, 4\,N)}{\mathrm{BesselI}(0, 4\,N)} - 3\,N^2 - \frac{1}{4}\,\mathrm{Bi}_w$$

```
> for Bi[w] from 0 by 0.1 to 0.5 do
> print (Bi[w],fsolve(Eq6=0,N=0.5..2));od;
```

0, 0.6547010314
0.1, 0.7427513447
0.2, 0.8275796835
0.3, 0.9122555293
0.4, 0.9981385727
0.5, 1.085789275

Example 3.12

```
> restart;
> Eq7:=3*N**2*(BesselI(2/3,2*N)/BesselI(-
1/3,2*N))**2+N*BesselI(2/3,2*N)/BesselI(-1/3,2*N)-3*N^2-
2/3*Bi[w];;
```

$$Eq7 := \frac{3\,N^2\,\mathrm{BesselI}\left(\frac{2}{3}, 2\,N\right)^2}{\mathrm{BesselI}\left(\frac{-1}{3}, 2\,N\right)^2} + \frac{N\,\mathrm{BesselI}\left(\frac{2}{3}, 2\,N\right)}{\mathrm{BesselI}\left(\frac{-1}{3}, 2\,N\right)} - 3\,N^2 - \frac{2}{3}\,\mathrm{Bi}_w$$

```
> for Bi[w] from 0 by 0.1 to 0.5 do
> print(Bi[w],fsolve(Eq7=0,N=0.5));od;
```

0, 0.9026377586
0.1, 0.9944089356
0.2, 1.081600252
0.3, 1.167465389
0.4, 1.253901495
0.5, 1.342180457

```
> restart;
> Eq8:=-1+sqrt(1+36*N**2)-27*N**2/(sqrt(1+36*N**2))-Bi[w];
```

$$Eq8 := -1 + \sqrt{1 + 36\,N^2} - \frac{27\,N^2}{\sqrt{1 + 36\,N^2}} - \mathrm{Bi}_w$$

```
> for Bi[w] from 0 by 0.1 to 0.5 do
> print(Bi[w],fsolve(Eq8=0,N=0.4..1.5));od;
```

0, 0.4714045208
0.1, 0.5688304372
0.2, 0.6560479087
0.3, 0.7379245314
0.4, 0.8164965809
0.5, 0.8928253913

3.2.2.4 Phase Change in Planar Geometry

The similarity technique is often employed to analyze planar conducted controlled phase change (freezing or melting) in a semi-infinite medium. In order to determine the location of the phase change front, the solution technique involves transcendental equations. The equations that require solution are [7, 8, 9, 10].

For the one-region Neumann problem (conduction in one phase only),

$$\sqrt{\pi}\lambda e^{\lambda^2} erf(\lambda) - S = 0 \tag{3.20}$$

For the two- region Neumann problem (conduction in both liquid and solid phases),

$$\frac{e^{-\lambda^2}}{\lambda erf\lambda} - \frac{ae^{-\lambda^2 b}}{\lambda erfc(\lambda\sqrt{b})} = \sqrt{\pi}/S \tag{3.21}$$

where a and b are thermal parameters.

For the three-region Neumann problem (a mushy zone between the liquid and solid phases),

$$\lambda e^{\lambda^2} erf\lambda + \frac{1}{2}c\sqrt{\pi}\left[e^{\lambda^2} erf\lambda\right]^2 - \frac{S}{\sqrt{\pi}} = 0 \tag{3.14}$$

where c is a thermal parameter.

The solution of Eq. (3.20) through (3.22) is demonstrated in Example 3.13.

Example 3.13

```
> restart:
> Eq1:=sqrt(Pi)*lambda*exp(lambda**2)*erf(lambda)-S;
```

$$Eq1 := \sqrt{\pi}\,\lambda\,e^{(\lambda^2)}\,\mathrm{erf}(\lambda) - S$$

```
> for S from 0.0 by 0.1 to 1.0 do
> print(S,fsolve(Eq1=0,lambda=1));od;
```

0., 0.

0.1, 0.2200162727

0.2, 0.3064239054

0.3, 0.3698802167

0.4, 0.4212378184
0.5, 0.4647859206
0.6, 0.5027615423
0.7, 0.5365121097
0.8, 0.5669254207
0.9, 0.5946235938
1.0, 0.6200626333

```
> restart:
> Eq2:=exp(-lambda**2)/(lambda*erf(lambda))-a*exp(-
lambda**2*b)/(lambda*erfc(lambda*sqrt(b)))-sqrt(Pi)/S;
```

$$Eq2 := \frac{e^{(-\lambda^2)}}{\lambda\,\mathrm{erf}(\lambda)} - \frac{a\,e^{(-\lambda^2\,b)}}{\lambda\,\mathrm{erfc}(\lambda\sqrt{b})} - \frac{\sqrt{\pi}}{S}$$

```
> a:=0.5;b:=2.0;
```

$$a := 0.5$$
$$b := 2.0$$

```
> Eq2;
```

$$\frac{e^{(-\lambda^2)}}{\lambda\,\mathrm{erf}(\lambda)} - \frac{0.5\,e^{(-2.0\,\lambda^2)}}{\lambda\,\mathrm{erfc}(1.414213562\,\lambda)} - \frac{\sqrt{\pi}}{S}$$

```
> for S from 0.1 by 0.1 to 1.0 do
> print(S,fsolve(Eq2=0,lambda=0.2));od;
```

0.1, 0.2024237566
0.2, 0.2699816008
0.3, 0.3147316165
0.4, 0.3479063904
0.5, 0.3739327492
0.6, 0.3950891269
0.7, 0.4127207430
0.8, 0.4276926201
0.9, 0.4405947067
1.0, 0.4518471635

```
> restart:
> Eq3:=lambda*exp(lambda**2)*erf(lambda)+1/2*c*sqrt(Pi)*
(exp(lambda**2)*erf(lambda))**2-S/sqrt(Pi);
```

$$Eq3 := \lambda\, e^{(\lambda^2)}\, \mathrm{erf}(\lambda) + \frac{1}{2}\, c\, \sqrt{\pi}\, \left(e^{(\lambda^2)}\right)^2\, \mathrm{erf}(\lambda)^2 - \frac{S}{\sqrt{\pi}}$$

```
> c:=1:
> for S from 0 by 1 to 5 do
> print(S,fsolve(Eq3=0,lambda=0..1));od;
```

0, 0.

1, 0.4505913611

2, 0.5898073674

3, 0.6788509059

4, 0.7440169189

5, 0.7951403395

3.3.2.5 Phase Change in Cylindrical Geometry

A similarity type of analysis for phase change in a cylindrical geometry leads to transcendental equations involving exponential integral functions [8,11]. The location of the phase change front when a sub cooled liquid freezes is given by

$$\lambda^2 e^{\lambda^2} Ei(-\lambda^2) + S = 0 \tag{3.23}$$

The solution of Eq. (3.23) is demonstrated in Example 3.14. Fig. 3.1 shows the graph of the solution.

Example 3.14

```
> restart:
> Eq1:=lambda**2*Ei(-lambda**2)*exp(lambda**2)+S=0;
```

$$Eq1 := \lambda^2\, \mathrm{Ei}(-\lambda^2)\, e^{(\lambda^2)} + S = 0$$

```
> for k from 1 to 9 do
```

```
> sol||k:=fsolve(subs(S=0.1*k,Eq1),lambda=0..5);od;
```

$$sol1 := 0.1846234866$$
$$sol2 := 0.3143246369$$
$$sol3 := 0.4490928787$$
$$sol4 := 0.6006191986$$
$$sol5 := 0.7810619641$$
$$sol6 := 1.009531614$$
$$sol7 := 1.323673905$$
$$sol8 := 1.817953217$$
$$sol9 := 2.858430025$$

```
> plot([seq([0.1*k,sol||k],k=1..9)],color=black);
```

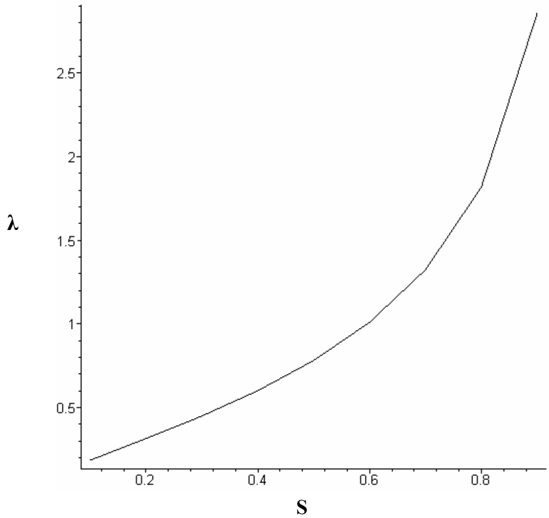

Figure 3.1 A graph of example 3.14

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Chapter 4

Elementary One-Dimensional Steady Conduction

4.1 INTRODUCTION

This chapter treats one-dimensional steady state heat conduction. It begins with a discussion of heat transfer in plane walls, hollow cylinders, hollow spheres, and truncated conical sections without heat generation. In each case, expressions for the temperature distribution and heat transfer are derived. The discussion is then extended to the analysis of composite systems using thermal networks. The chapter concludes with an analysis of systems where the heat conduction is accompanied by a uniform heat generation. The equations appearing in this chapter can either be found in undergraduate heat transfer texts or created from a basic knowledge of heat transfer. Consequently, no references are cited in this chapter.

4.2 THE PLANE WALL

Consider a plane wall of thickness, L, made of a material with a thermal conductivity, k, shown in Fig. 4.1. The left face of the wall at $x = 0$ receives heat by convection from a hot fluid at temperature $T_{\infty,1}$ while the right face is cooled by a fluid at $T_{\infty,2}$. The convective heat transfer coefficients at the hot and cold faces of the wall are h_1 and h_2 respectively. For steady state convection with no internal heat generation and constant thermal conductivity, the differential equation governing the temperature distribution in the wall is

$$\frac{d^2T}{dx^2} = 0 \tag{4.1}$$

The boundary conditions describing the convection processes on two faces of the wall can be written as

$$at\ x=0 \qquad h_1(T-T_{\infty,1})-k\frac{dT}{dx}=0 \tag{4.2}$$

$$at\ x=L \qquad h_2(T-T_{\infty,2})-k\frac{dT}{dx}=0 \tag{4.3}$$

The solution for the temperature distribution and heat transfer in the plane wall is presented in Example 4.1.

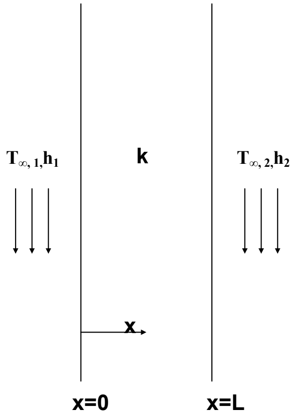


Figure 4.1: The Plane Wall

Example 4.1:

```
> restart;
> Eq1:=diff(T(x),x,x)=0;
```

$$Eq1 := \frac{d^2}{dx^2}\mathrm{T}(x) = 0$$

```
> Eq2:=dsolve(Eq1,T(x));
```

$$Eq2 := \mathrm{T}(x) = \_C1\, x + \_C2$$

```
> Eq3:=diff(Eq2,x);
```

$$Eq3 := \frac{d}{dx}\mathrm{T}(x) = \_C1$$

```
> bc1:=h[1]*(subs(x=0,rhs(Eq2))-T[infinity1])-
k*subs(x=0,rhs(Eq3));
```

$$bc1 := h_1\,(\_C2 - T_{\infty 1}) - k\,\_C1$$

```
> bc2:=h[2]*(subs(x=L,rhs(Eq2))-
T[infinity2])+k*subs(x=L,rhs(Eq3));
```

$$bc2 := h_2\,(\_C1\,L + \_C2 - T_{\infty 2}) + k\,\_C1$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \left\{\_C2 = \frac{k\,h_2\,T_{\infty 2} + h_2\,L\,h_1\,T_{\infty 1} + k\,h_1\,T_{\infty 1}}{k\,h_2 + h_2\,L\,h_1 + k\,h_1},\ \_C1 = -\frac{h_1\,h_2\,(-T_{\infty 2} + T_{\infty 1})}{k\,h_2 + h_2\,L\,h_1 + k\,h_1}\right\}$$

```
> assign(const);
```

```
> Eq4:=Eq2;
```

Temperature distribution in the wall

$$Eq4 := \mathrm{T}(x) = -\frac{h_1\,h_2\,(-T_{\infty 2} + T_{\infty 1})\,x}{k\,h_2 + h_2\,L\,h_1 + k\,h_1} + \frac{k\,h_2\,T_{\infty 2} + h_2\,L\,h_1\,T_{\infty 1} + k\,h_1\,T_{\infty 1}}{k\,h_2 + h_2\,L\,h_1 + k\,h_1}$$

```
> T[s1]:=subs(x=0,rhs(Eq4));
```

Temperature on the left face of the wall

$$T_{s1} := \frac{k\,h_2\,T_{\infty 2} + h_2\,L\,h_1\,T_{\infty 1} + k\,h_1\,T_{\infty 1}}{k\,h_2 + h_2\,L\,h_1 + k\,h_1}$$

```
> T[s2]:=subs(x=L,rhs(Eq4));
```

Temperature on the right face of the wall

$$T_{s2} := -\frac{h_1\,h_2\,(-T_{\infty 2} + T_{\infty 1})\,L}{k\,h_2 + h_2\,L\,h_1 + k\,h_1} + \frac{k\,h_2\,T_{\infty 2} + h_2\,L\,h_1\,T_{\infty 1} + k\,h_1\,T_{\infty 1}}{k\,h_2 + h_2\,L\,h_1 + k\,h_1}$$

```
> Heat_flux=simplify(h[1]*(T[infinity1]-T[s1]));
```

$$Heat\_flux = \frac{h_1\, k\, h_2\, (-T_{\infty 2} + T_{\infty 1})}{k\, h_2 + h_2\, L\, h_1 + k\, h_1}$$

The case of fixed temperature boundary conditions can be derived from the prior solutions by letting

$$h_1 \to \infty \quad and \quad h_2 \to \infty$$

4.3 HOLLOW CYLINDER

Consider a hollow cylinder of inside radius, r_1, outside radius, r_2, and length L, made of a material with a thermal conductivity, k, as shown in Figure 4.2. The inner surface is heated by convection by a fluid at $T_{\infty,1}$ and convection coefficient h_1. The outer surface loses heat by convection to the surroundings at temperature, $T_{\infty,2}$ with a heat conduction coefficient of h_2. For steady state conduction in the radial direction with no internal heat generation and constant thermal conductivity, the appropriate differential equation is

$$\frac{d}{dr}(r\frac{dT}{dr}) = 0 \tag{4.4}$$

The convection process on the inside and outside surfaces can be described in terms of the boundary conditions

$$r = r_1, h_1(T - T_{\infty,1}) - k\frac{dT}{dr} = 0 \tag{4.5}$$

$$r = r_2, h_2(T - T_{\infty,2}) - k\frac{dT}{dr} = 0 \tag{4.6}$$

Example 4.2 shows how *Maple* can be used to determine the temperature distribution and heat transfer rate in the hollow cylinder. The inside and outside surface temperatures are also determined.

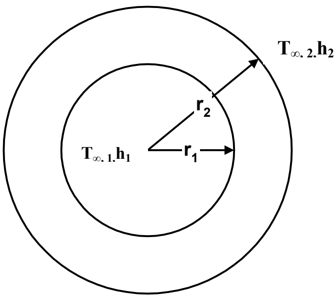


Figure 4.2: The hollow cylinder.

Example 4.2:

```
> restart;
```

```
> Eq1:=diff(r*diff(T(r),r),r)=0;
```

$$Eq1 := \left(\frac{d}{dr}\mathrm{T}(r)\right) + r\left(\frac{d^2}{dr^2}\mathrm{T}(r)\right) = 0$$

```
> Eq2:=dsolve(Eq1,T(r));
```

$$Eq2 := \mathrm{T}(r) = \_C1 + \_C2\ \ln(r)$$

```
> Eq3:=diff(Eq2,r);
```

$$Eq3 := \frac{d}{dr}\mathrm{T}(r) = \frac{\_C2}{r}$$

```
> bc1:=h[1]*(subs(r=ri,rhs(Eq2))-T[infinity1])-
k*subs(r=ri,rhs(Eq3));
```

$$bc1 := h_1\,(\_C1 + \_C2\ \ln(ri) - T_{\infty 1}) - \frac{k\ \_C2}{ri}$$

```
> bc2:=h[2]*(subs(r=ro,rhs(Eq2))-
T[infinity2])+k*subs(r=ro,rhs(Eq3));
```

$$bc2 := h_2\,(\_C1 + \_C2\ \ln(ro) - T_{\infty 2}) + \frac{k\ \_C2}{ro}$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \{\_C2 = \frac{h_2\, ro\, h_1\, ri\, (-T_{\infty 2} + T_{\infty 1})}{h_2\, ro\, h_1\, ri\, \ln(ri) - h_2\, ro\, k - h_1\, ri\, h_2\, ro\, \ln(ro) - h_1\, ri\, k},$$

$$\_C1 = -\frac{h_1\, ri\, h_2\, ro\, \ln(ro)\, T_{\infty 1} + h_1\, ri\, k\, T_{\infty 1} - T_{\infty 2}\, h_2\, ro\, h_1\, ri\, \ln(ri) + T_{\infty 2}\, h_2\, ro\, k}{h_2\, ro\, h_1\, ri\, \ln(ri) - h_2\, ro\, k - h_1\, ri\, h_2\, ro\, \ln(ro) - h_1\, ri\, k}\}$$

```
> assign(const);
```

```
> Eq4:=simplify(Eq2);
```

Temperature distribution in the hollow cylinder

$$Eq4 := \mathrm{T}(r) = -(h_1\, ri\, h_2\, ro\, \ln(ro)\, T_{\infty 1} + h_1\, ri\, k\, T_{\infty 1} - T_{\infty 2}\, h_2\, ro\, h_1\, ri\, \ln(ri) + T_{\infty 2}\, h_2\, ro\, k + h_2\, ro\, h_1\, ri\, \ln(r)\, T_{\infty 2} - h_2\, ro\, h_1\, ri\, \ln(r)\, T_{\infty 1}) \Big/ (h_2\, ro\, h_1\, ri\, \ln(ri) - h_2\, ro\, k - h_1\, ri\, h_2\, ro\, \ln(ro) - h_1\, ri\, k)$$

```
> T[s1]:=subs(r=ri,rhs(Eq4));
```

Temperature on the inside surface

$$T_{s1} := -\frac{h_1\, ri\, h_2\, ro\, \ln(ro)\, T_{\infty 1} + h_1\, ri\, k\, T_{\infty 1} + T_{\infty 2}\, h_2\, ro\, k - h_2\, ro\, h_1\, ri\, \ln(ri)\, T_{\infty 1}}{h_2\, ro\, h_1\, ri\, \ln(ri) - h_2\, ro\, k - h_1\, ri\, h_2\, ro\, \ln(ro) - h_1\, ri\, k}$$

```
> T[s2]:=subs(r=ro,rhs(Eq4));
```

Temperature on the outside surface

$$T_{s2} := -\frac{h_1\, ri\, k\, T_{\infty 1} - T_{\infty 2}\, h_2\, ro\, h_1\, ri\, \ln(ri) + T_{\infty 2}\, h_2\, ro\, k + h_2\, ro\, h_1\, ri\, \ln(ro)\, T_{\infty 2}}{h_2\, ro\, h_1\, ri\, \ln(ri) - h_2\, ro\, k - h_1\, ri\, h_2\, ro\, \ln(ro) - h_1\, ri\, k}$$

```
> q:=simplify(h[1]*(T[infinity1]-T[s1])*2*Pi*ri*L);
```

Heat transfer rate

$$q := -\frac{2\, h_1\, h_2\, ro\, k\, (-T_{\infty 2} + T_{\infty 1})\, \pi\, ri\, L}{h_2\, ro\, h_1\, ri\, \ln(ri) - h_2\, ro\, k - h_1\, ri\, h_2\, ro\, \ln(ro) - h_1\, ri\, k}$$

The case of fixed temperature boundary conditions can be derived from the prior solutions by letting

$$h_1 \to \infty \quad and \quad h_2 \to \infty$$

and replacing $T_{\infty,1}$ with T_{s1} and $T_{\infty,2}$ with T_{s2}.

4.4 THE HOLLOW SPHERE

The description in section 4.3 is also applicable to a hollow sphere. In this case the appropriate differential equation is

$$\frac{d}{dr}(r^2\frac{dT}{dr}) = 0 \tag{4.7}$$

Equations (4.5) and (4.6) also serve as the boundary conditions for the hollow sphere and the solutions for the temperature distribution and the heat transfer rate are presented in Example 4.3.

Example 4.3:

```
> restart; Eq1:=diff(r^2*diff(T(r),r),r)=0;
```

$$Eq1 := 2\,r\left(\frac{d}{dr}\mathrm{T}(r)\right) + r^2\left(\frac{d^2}{dr^2}\mathrm{T}(r)\right) = 0$$

```
> Eq2:=dsolve(Eq1,T(r));
```

$$Eq2 := \mathrm{T}(r) = \_C1 + \frac{\_C2}{r}$$

```
> Eq3:=diff(Eq2,r);
```

$$Eq3 := \frac{d}{dr}\mathrm{T}(r) = -\frac{\_C2}{r^2}$$

```
> bc1:=h[1]*(subs(r=ri,rhs(Eq2))-T[infinity1])-
k*subs(r=ri,rhs(Eq3));
```

$$bc1 := h_1\left(\_C1 + \frac{\_C2}{ri} - T_{\infty 1}\right) + \frac{k\,\_C2}{ri^2}$$

```
> bc2:=h[2]*(subs(r=ro,rhs(Eq2))-
T[infinity2])+k*subs(r=ro,rhs(Eq3));
```

$$bc2 := h_2\left(\_C1 + \frac{\_C2}{ro} - T_{\infty 2}\right) - \frac{k\,\_C2}{ro^2}$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \left\{ \_C2 = \frac{h_1\, ri^2\, h_2\, ro^2\, (T_{\infty 1} - T_{\infty 2})}{-h_1\, ri^2\, h_2\, ro + h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri + h_2\, ro^2\, k}, \right.$$

$$\left. \_C1 = \frac{h_2\, ro^2\, h_1\, ri\, T_{\infty 2} + h_2\, ro^2\, k\, T_{\infty 2} - T_{\infty 1}\, h_1\, ri^2\, h_2\, ro + T_{\infty 1}\, h_1\, ri^2\, k}{-h_1\, ri^2\, h_2\, ro + h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri + h_2\, ro^2\, k} \right\}$$

```
> assign(const):
> Eq4:=simplify(Eq2);
```

(Temperature distribution in the hollow sphere)

$$Eq4 := \mathrm{T}(r) = (r\, h_2\, ro^2\, h_1\, ri\, T_{\infty 2} + r\, h_2\, ro^2\, k\, T_{\infty 2} - r\, T_{\infty 1}\, h_1\, ri^2\, h_2\, ro + r\, T_{\infty 1}\, h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri^2\, T_{\infty 1} - h_1\, ri^2\, h_2\, ro^2\, T_{\infty 2}) \Big/ ((-h_1\, ri^2\, h_2\, ro + h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri + h_2\, ro^2\, k)\, r)$$

```
> T[s1]:=subs(r=ri,rhs(Eq4));
```

(Temperature on the inside surface)

$$T_{s1} := \frac{ri\, h_2\, ro^2\, k\, T_{\infty 2} - ri^3\, T_{\infty 1}\, h_1\, h_2\, ro + ri^3\, T_{\infty 1}\, h_1\, k + h_2\, ro^2\, h_1\, ri^2\, T_{\infty 1}}{(-h_1\, ri^2\, h_2\, ro + h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri + h_2\, ro^2\, k)\, ri}$$

```
> T[s2]:=subs(r=ro,rhs(Eq4));
```

(Temperature on the outside surface)

$$T_{s2} := \frac{ro^3\, h_2\, h_1\, ri\, T_{\infty 2} + ro^3\, h_2\, k\, T_{\infty 2} + ro\, T_{\infty 1}\, h_1\, ri^2\, k - h_1\, ri^2\, h_2\, ro^2\, T_{\infty 2}}{(-h_1\, ri^2\, h_2\, ro + h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri + h_2\, ro^2\, k)\, ro}$$

```
> q:=simplify(h[1]*(T[infinity1]-T[s1])*4*Pi*ri^2);
```

(Heat transfer rate)

$$q := \frac{4\, h_1\, h_2\, ro^2\, k\, (T_{\infty 1} - T_{\infty 2})\, \pi\, ri^2}{-h_1\, ri^2\, h_2\, ro + h_1\, ri^2\, k + h_2\, ro^2\, h_1\, ri + h_2\, ro^2\, k}$$

The case of fixed temperature boundary conditions can be derived from the prior solutions by letting

$$h_1 \to \infty \quad and \quad h_2 \to \infty$$

and replacing $T_{\infty,1}$ with T_{s1} and $T_{\infty,2}$ with T_{s2}.

4.5 TRUNCATED CONICAL SECTIONS

In Figs. 4.3a and 4.3b, the cross sectional area, $A(x)$, is a function of x

$$A(x) = \frac{\pi}{4} d(x)^2 \tag{4.8}$$

where the diameter, $d(x) = ax^{1/2}$, applies for the configuration of Fig. 4.3a and $d(x) = ax^{3/2}$ applies for the configuration of Fig 4.3b, with a being a numerical constant. For one dimensional steady state heat conduction with no internal generation, the governing differential equation can be written as

$$\frac{d}{dx}\left[A(x)\frac{dT}{dx}\right] = 0 \tag{4.9}$$

with the boundary conditions of

$$x = x_1 \qquad T = T_1 \tag{4.10}$$

$$x = x_2 \qquad T = T_2 \tag{4.11}$$

derived from the assumption that the temperature on the faces are T_1 and T_2, respectively. Example 4.4 derives the solutions for the temperature distribution and heat transfer rates for the configurations shown in Figs. 4.3a and 4.3b.

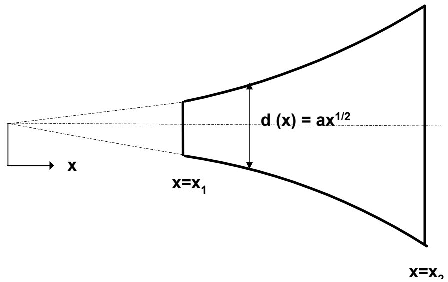


Fig 4.3a

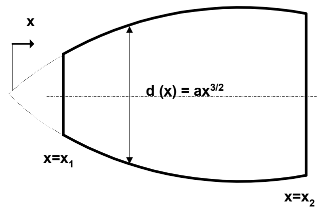


Fig 4.3b

Example 4.4:

```
> restart:
> Eq1:=diff(A(x)*diff(T(x),x),x)=0;
```

$$Eq1 := \left(\frac{d}{dx}\mathrm{A}(x)\right)\left(\frac{d}{dx}\mathrm{T}(x)\right) + \mathrm{A}(x)\left(\frac{d^2}{dx^2}\mathrm{T}(x)\right) = 0$$

```
> D(x):=a*sqrt(x); A(x):=Pi*D(x)^2/4;
```

$$\mathrm{D}(x) := a\sqrt{x}$$

$$\mathrm{A}(x) := \frac{\pi a^2 x}{4}$$

This is the configuration of Fig 4.3a.

```
> Eq2:=dsolve({Eq1,T(x1)=T[1],T(x2)=T[2]},T(x));
```

$$Eq2 := \mathrm{T}(x) = \frac{\ln(x2)\,T_1 - T_2\ln(x1)}{\ln(x2) - \ln(x1)} - \frac{(-T_2 + T_1)\ln(x)}{\ln(x2) - \ln(x1)}$$

```
> Eq3:=diff(%,x);
```

$$Eq3 := \frac{d}{dx}\mathrm{T}(x) = -\frac{-T_2 + T_1}{(\ln(x2) - \ln(x1))\,x}$$

```
> q:=-k*A(x)*rhs(Eq3);
```

$$q := \frac{1}{4}\,\frac{k\,\pi\,a^2\,(-T_2 + T_1)}{\ln(x2) - \ln(x1)}$$

As a numerical example, consider a conical section fabricated from pure aluminum, k=236 W/m-K, with the smaller face located at x = 25 mm and the larger face located at x = 125 mm. Let T_1 = 600 K and T_2 = 400 K. Assume that the diameter variation with x is characterized by the parameter a = 0.5 m.

```
> x1:=0.025: x2:=0.125: k:=236: a:=0.5: T[1]:=600:
T[2]:=400:
> Temp_distribution:=rhs(Eq2);
```

$$Temp\_distribution := 141.5940654 - 124.2669869\ln(x)$$

```
> Heat_transfer_rate_in_Watts:=evalf(q);
```

$$Heat\_transfer\_rate\_in\_Watts := 5758.344738$$

```
> restart; Eq1:=diff(A(x)*diff(T(x),x),x)=0;
```

$$Eq1 := \left(\frac{d}{dx}\mathrm{A}(x)\right)\left(\frac{d}{dx}\mathrm{T}(x)\right) + \mathrm{A}(x)\left(\frac{d^2}{dx^2}\mathrm{T}(x)\right) = 0$$

```
> D(x):=a*x^(3/2); A(x):=Pi*D(x)^2/4;
```

$$\mathrm{D}(x) := a\,x^{(3/2)}$$

$$\mathrm{A}(x) := \frac{\pi\, a^2\, x^3}{4}$$

This is the configuration of Fig. 4.3b.

```
> Eq2:=simplify(dsolve({Eq1,T(x1)=T[1],T(x2)=T[2]},T(x)));
```

Temperature Distribution

$$Eq2 := \mathrm{T}(x) = \frac{-x^2\, T_2\, x2^2 + x^2\, T_1\, x1^2 - x1^2\, x2^2\, T_1 + x1^2\, x2^2\, T_2}{(x1^2 - x2^2)\, x^2}$$

```
> Eq3:=diff(%,x);
```

$$Eq3 := \frac{d}{dx}\mathrm{T}(x) = \frac{-2\, x\, T_2\, x2^2 + 2\, x\, T_1\, x1^2}{(x1^2 - x2^2)\, x^2} - \frac{2\,(-x^2\, T_2\, x2^2 + x^2\, T_1\, x1^2 - x1^2\, x2^2\, T_1 + x1^2\, x2^2\, T_2)}{(x1^2 - x2^2)\, x^3}$$

```
> q:=simplify(-k*A(x)*rhs(Eq3));
```

Heat transfer rate

$$q := -\frac{1}{2}\,\frac{k\,\pi\, a^2\, x2^2\, x1^2\,(T_1 - T_2)}{x1^2 - x2^2}$$

As another numerical example, consider a conical section fabricated from pure aluminum, k = 238 W/m-K, with the smaller of two faces located at x = 75 mm. The larger face is located at x = 225mm. Let T_1 = 100 K and T_2 = 20 K. Assume that the diameter variation with x is characterized by a = 1.0 m. The solution is plotted in Fig. 4.4.

```
> x1:=0.075: x2:=0.225: k:=238: a:=1: T[1]:=100: T[2]:=20:
```

```
> Temp_distribution:=evalf(rhs(Eq2),3);
Heat_Transfer_rate_in_Watts:=evalf(q,3);
```

$$Temp\_distribution := -\frac{22.2\,(-0.448\,x^2 - 0.0227)}{x^2}$$

$$Heat\_Transfer\_rate\_in\_Watts := 188.$$

```
> plot(Temp_distribution,x=0.075..0.225,color=black,
labels=["Length (m)","Temperature (K)"],title="Temperature
as a Function of Distance");
```

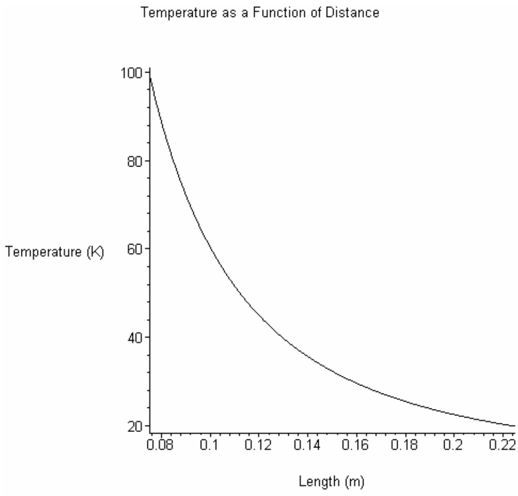
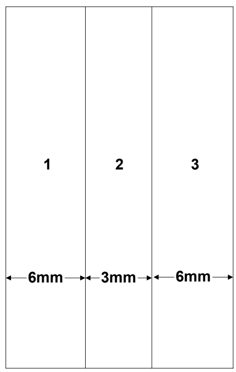


Figure 4.4: Plot of temperature as a function of cone length.

4.6 COMPOSITE SECTIONS

Heat transfer through composite systems (planar, cylindrical, and spherical) is best computed using the concept of thermal resistance. Three examples, one for each geometry, are presented in order to show how *Maple* can be used to study the thermal behavior of some practical systems.



1 & 3 - glass, 2 - air

Fig. 4.5: Composite plane configuration, double pane window

4.6.1 Composite Plane System

Example 4.5:
A double pane window (Fig 4.5) consists of two pieces of glass 6mm thick separated by an air gap of 3mm. A triple pane (Fig 4.6) design uses three pieces of glass 6mm thick separated by two air gaps of 3mm. Both designs are to be evaluated for installation in a room at 20°C with outside ambient air at -10°C. The natural convection coefficient on the inside surface of the window is estimated to be 12W/m^2-K. The convection coefficient associated with the outside environment is expected to vary between 10 to 100 W/m^2-K, depending on the wind velocity. Compare the heat loss from each window design as a function of the outside convection coefficient assuming

the window surface area to be 0.4 m^2, the thermal conductivity of the air layer, k_a = 0.0245 W/m-K and the thermal conductivity of the glass, k_g = 1.40 W/m-K.

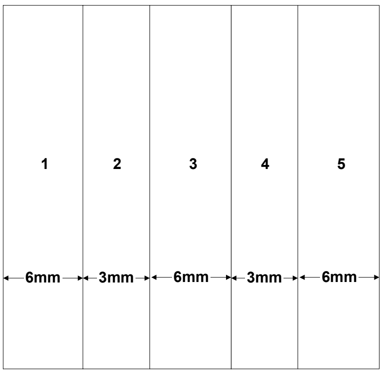


Fig. 4.6: Composite plane configuration; triple pane window.

The physical arrangements are shown in figs 4.5 and 4.6. The thermal resistances are denoted as R_2 (twp-pane) and R_3 (three-pane).

```
> restart;
```

```
> R[2]:=1/(h[i]*A)+2*L[g]/(k[g]*A)+L[a]/(k[a]*A)+1/(h[o]*A);
```

$$R_2 := \frac{1}{h_i A} + \frac{2 L_g}{k_g A} + \frac{L_a}{k_a A} + \frac{1}{h_o A}$$

```
> R[3]:=1/(h[i]*A)+3*L[g]/(k[g]*A)+2*L[a]/(k[a]*A)+1/(h[o]*
A);
```

$$R_3 := \frac{1}{h_i A} + \frac{3 L_g}{k_g A} + \frac{2 L_a}{k_a A} + \frac{1}{h_o A}$$

```
> q[2]:=(T[infinity,i]-T[infinity,o])/R[2];
```

$$q_2 := \frac{T_{\infty,i} - T_{\infty,o}}{\frac{1}{h_i A} + \frac{2 L_g}{k_g A} + \frac{L_a}{k_a A} + \frac{1}{h_o A}}$$

```
> q[3]:=(T[infinity,i]-T[infinity,o])/R[3];
```

$$q_3 := \frac{T_{\infty,i} - T_{\infty,o}}{\frac{1}{h_i A} + \frac{3 L_g}{k_g A} + \frac{2 L_a}{k_a A} + \frac{1}{h_o A}}$$

```
> T[infinity,i]:=20;T[infinity,o]:=-10;h[i]:=12;A:=0.4;
L[g]:=0.006; L[a]:=0.003; k[g]:=1.4;k[a]:=0.0245;
```

$$T_{\infty,i} := 20$$

$$T_{\infty,o} := -10$$

$$h_i := 12$$

$$A := 0.4$$

$$L_g := 0.006$$

$$L_a := 0.003$$

$$k_g := 1.4$$

$$k_a := 0.0245$$

```
> plot([q[2],q[3]],h[o]=10..100,color=black,legend=[q2,q3],
linestyle=[SOLID,DASH],labels=["Outside Air Heat Transfer
Coefficent (W/m-K)","Heat Transfer (W)"]);
```

(*Maple* plot on next page)

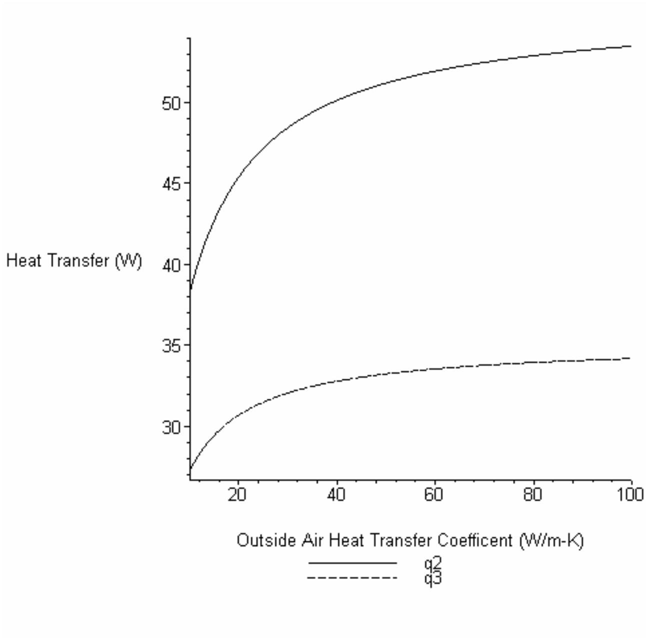


Fig. 4.7: Heat loss through two-pane and three-pane windows as a function of outside air heat transfer coefficient.

The graphical results (Fig 4.7) indicate that the triple-pane design is more effective as a heat loss deterrent compared with the double-pane design. This is due to the increase in the overall thermal resistance introduced by the third pane and the second air gap. At h_o = 10W/m^2-K the reduction in heat loss with the triple-pane window compared with that of the double-pane window is 28.7 percent. This jumps to 36.1 percent at h_o = 100W/m^2-K. Figure 4.7 shows that at low values of h_o, the heat loss is significantly effected by changes in h_o. However, as h_o increases, that is, as the outside convection resistance decreases, the heat loss curves tend to flatten indicating that the influence of h_o on the heat loss is considerably reduced. This is particularly true of the triple-pane window.

4.6.2 Composite Cylindrical System

Example 4.6:
A long metallic tube, shown in Fig. 4.8, is heated by wrapping a thin electrical tape around its outer surface. The tube has inner and outer radii of 20mm and 30mm respectively. The inner surface of the tube is maintained at a temperature of 15°C while the outside of the tube is exposed to ambient air at 10°C with a convection coefficient of 80W/m^2-K. The electrical power required to maintain the heater temperature at 30°C depends on the thermal conductivity of the tube, k, and the contact resistance between the heater and the outside surface of the tube , $R_{t,c}$. Study how the variation of the thermal conductivity and their contact resistance affect the total heat flow into the tube and the total heater power required. Assume that $25 \le k \le 200$ (W/m-K) and that $0.1 \le R_{t,c} \le 0.05$ (°C/W).

Denote the heat flow into the tube by q_i (W/m), the heat flow to the ambient by q_o (W/M) and the total heater power by q_t (W/m). Observe that $q_t = q_o + q_i$

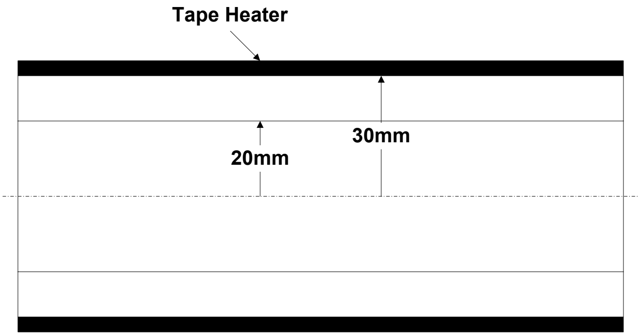


Fig. 4.8: Hollow cylinder with electrical heater

```
> restart;
```

```
> R[i]:=R[t,c]+ln(r[o]/r[i])/(2*Pi*k);
```

$$R_i := R_{t,c} + \frac{1}{2}\frac{\ln\left(\frac{r_o}{r_i}\right)}{\pi k}$$

```
> R[o]:=1/(h[o]*2*Pi*r[o]);
```

$$R_o := \frac{1}{2}\frac{1}{h_o\,\pi\,r_o}$$

```
> q[i]:=(T[h]-T[i])/R[i];
```

$$q_i := \frac{T_h - T_i}{R_{t,c} + \frac{1}{2}\frac{\ln\left(\frac{r_o}{r_i}\right)}{\pi\,k}}$$

```
> q[o]:=(T[h]-T[infinity])/R[o];
```

$$q_o := 2\,(T_h - T_\infty)\,h_o\,\pi\,r_o$$

```
> q[t]:=q[i]+q[o];
```

$$q_t := \frac{T_h - T_i}{R_{t,c} + \frac{1}{2}\frac{\ln\left(\frac{r_o}{r_i}\right)}{\pi\,k}} + 2\,(T_h - T_\infty)\,h_o\,\pi\,r_o$$

>r[i]:=0.02;r[o]:=0.03;h[o]:=80;T[i]:=15;T[infinity]:=10;T[h]:=30;

$$r_i := 0.02$$

$$r_o := 0.03$$

$$h_o := 80$$

$$T_i := 15$$

$$T_\infty := 10$$

$$T_h := 30$$

```
> for n from 1 to 5 do
qinside||n:=subs(R[t,c]=n/100,q[i]);od:

> plot([qinside||(1..5)],k=25..200,y=0..2000,color=black,
linestyle=[DASH,SOLID,DASH,DOT,SOLID],thickness=[0,1,2,0,2],
legend=["R[t,c] = 0.01","R[t,c] = 0.02","R[t,c] = 0.03",
```

```
"R[t,c] = 0.04","R[t,c] = 0.05"],labels=["Conduction
Coefficient (W/m-K)", "Heat Transfer (W/m)"]);
```

```
> qtotal[m]:=subs(R[t,c]=m/100,q[t]);
```

$$qtotal_m := \frac{15}{\frac{m}{100} + \frac{0.2027325540}{\pi\, k}} + 96.00\,\pi$$

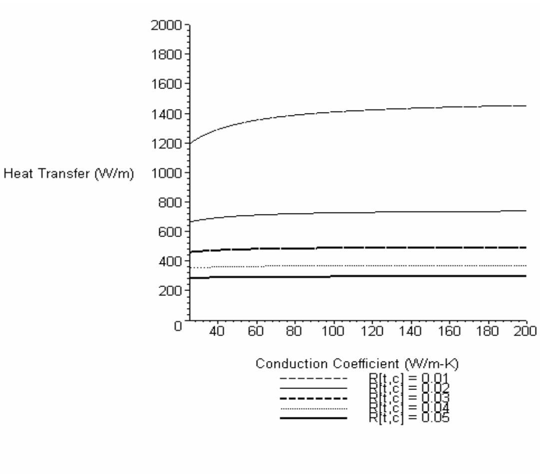


Fig 4.9: Heat flow into tube

```
> for j from 1 to 5 do qtotal||j:=subs(m=j,qtotal[m]);od;
```

$$qtotal1 := \frac{15}{\frac{1}{100} + \frac{0.2027325540}{\pi\, k}} + 96.00\,\pi$$

$$qtotal2 := \frac{15}{\frac{1}{50} + \frac{0.2027325540}{\pi\, k}} + 96.00\,\pi$$

$$qtotal3 := \frac{15}{\frac{3}{100} + \frac{0.2027325540}{\pi k}} + 96.00\,\pi$$

$$qtotal4 := \frac{15}{\frac{1}{25} + \frac{0.2027325540}{\pi k}} + 96.00\,\pi$$

$$qtotal5 := \frac{15}{\frac{1}{20} + \frac{0.2027325540}{\pi k}} + 96.00\,\pi$$

```
> plot([qtotal1,qtotal2,qtotal3,qtotal4,qtotal5],k=25..200,
y=0..2000,color=black,linestyle=[SOLID,DASH,DOT,SOLID,DASH],
legend=["R[t,c] = 0.01","R[t,c] = 0.02","R[t,c] = 0.03",
"R[t,c] = 0.04","R[t,c] = 0.05"],labels=["Conduction
Coefficient (W/m-K)","Heat Transfer (W/m)"]);
```

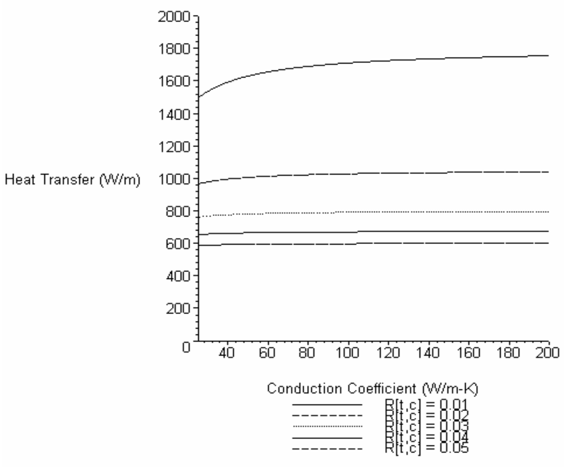


Fig. 4.10: Total Heat Transfer

Because h_o was kept fixed, q_o is fixed at 310.6 W/m. Fig. 4.9 shows the heat flow into the tube as a function of k for $R_{t,c}$ = 0.01, 0.02, 0.03, 0.04, and 0.05 °C/W. As expected, q_i increases as k increases and decreases as $R_{t,c}$ increases. Because q_o is fixed, the curves for q_t shown in Fig 4.10 follow the pattern of Fig 4.9.

4.6.3 Composite Spherical System

Example 4.7:
A composite spherical tank consisting of an inner shell of lead and an outer shell of stainless steel, shown in Fig. 4.11, is to be designed for storing radioactive material and with the capability of removing a heat specification of 32,725 W. The outer surface of the tank is to be cooled by exposing it to a stream of water at 10°C. Control of the water flow allows the convection heat transfer coefficient to be regulated in the range from 100 to 1000 W/m^2-K. The inside and outside radii of the lead shell are 0.25 m and 0.30 m respectively. The outer radius of the stainless steel shell can range from 0.30 to 0.50 m. It is desired to investigate how the maximum lead temperature, which occurs on its inside surface, is affected by variations in h and the outer radius of the stainless steel tube.

What minimum value of h must be maintained with the stainless steel shell of thickness 0.05mm to ensure that the maximum lead temperature does not exceed 225°C? Assume the thermal conductivities of lead and steel to be 35 and 15 W/m-K respectively.

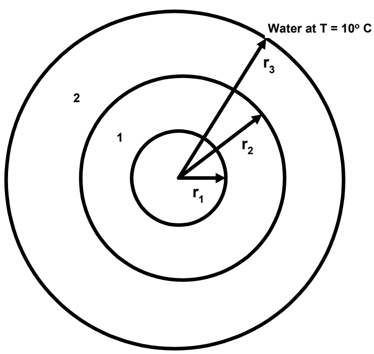


Fig. 4.11: Composite hollow sphere. r_1 = 0.25m , r_2 = 0.30m, r_3 = 0.30- 0.50m, 1 – lead , 2 – stainless steel.

```
> restart;
> R:=1/(4*Pi*k[1])*(1/r[1]-1/r[2])+1/(4*Pi*k[s])*(1/r[2]-
1/r[3])+1/(4*Pi*r[3]^2*h);
```

$$R := \frac{1}{4}\,\frac{\frac{1}{r_1}-\frac{1}{r_2}}{\pi\,k_1}+\frac{1}{4}\,\frac{\frac{1}{r_2}-\frac{1}{r_3}}{\pi\,k_s}+\frac{1}{4}\,\frac{1}{\pi\,r_3^{\,2}\,h}$$

```
> Tmax:=T[infinity]+q*R;
```

$$Tmax := T_\infty + q\left(\frac{1}{4}\,\frac{\frac{1}{r_1}-\frac{1}{r_2}}{\pi\,k_1}+\frac{1}{4}\,\frac{\frac{1}{r_2}-\frac{1}{r_3}}{\pi\,k_s}+\frac{1}{4}\,\frac{1}{\pi\,r_3^{\,2}\,h}\right)$$

```
> r[1]:=0.25;r[2]:=0.30;k[1]:=35;k[s]:=15;T[infinity]:=10;
q:=32725;
```

$$r_1 := 0.25$$

$$r_2 := 0.30$$

$$k_1 := 35$$

$$k_s := 15$$

$$T_\infty := 10$$

$$q := 32725$$

```
> for n from 0 to 4 do
Tmax||n:=subs(r[3]=0.30+0.05*n,Tmax);od:

> plot([Tmax||(0..4)],h=100..1000,y=0..400,color=black,
legend=["R[3] = 0.30 m (No Stainless Steel Shell)","R[3] =
0.35 m","R[3] = 0.40 m","R[3] = 0.45 m","R[3] = 0.50 m"],
labels=["Convection Coefficient 'h' (W/m-K)","Tmax (ºC)"],
linestyle=[DOT,DASH,SOLID,DASH,SOLID]);
```

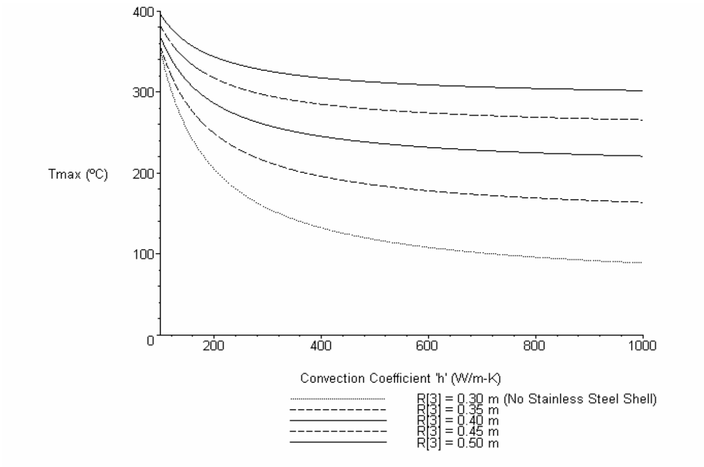


Fig. 4.12: Maximum lead temperature

Figure 4.12 shows how the maximum lead temperature, T_{max}, is affected by variations in h and r_3. For a given stainless steel shell thickness, T_{max} decreases as h increases (the convective heat transport increases). The lowest value of T_{max} occurs when the steel shell is completely removed ($r_3 = 0.30$ m). However, for the system to operate reliably, a certain minimum steel thickness is desired.

For a stainless steel shell of 0.05 m thickness, r = 0.35 m and the maximum lead temperature can be designated by T_{max1}. We can use the solve command to determine the value of h corresponding to
T_{max1}=220 ºC.

```
> h[min]:=solve(Tmax1=220,h);
```

$$h_{min} := 273.5114041$$

4.7 HEAT CONDUCTION WITH UNIFORM HEAT GENERATION

Heat conduction is sometimes accompanied by heat generation within the medium. For example, the energy generation may be due to the passage of electric current, an exothermic chemical reaction, the absorption of electromagnetic radiation, or nuclear activity within the medium. This section considers situations in which the heat generation occurs uniformly throughout the medium. More complex problems in which the heat generation is location or temperature dependent will be treated in chapter 5.

4.7.1 The Plane Wall

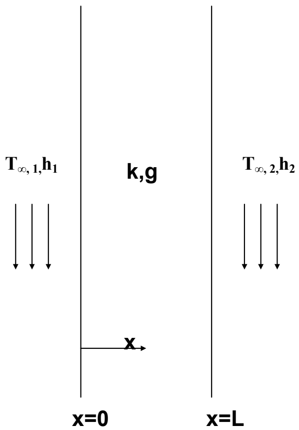


Fig 4.13: Plane wall with uniform heat generation

Consider the plane wall of Fig. 4.13 in which there is uniform heat generation denoted by g (W/m^3). The wall has a thickness, L and a thermal conductivity, k. The left face of the wall at x = 0 is cooled by a fluid at temperature $T_{\infty,1}$ with convective heat transfer coefficient, h_1. The right face of the wall at x = L is cooled by a fluid at temperature $T_{\infty,2}$ with a convective heat transfer coefficient, h_2. The unknown surface temperatures are $T_{s,1}$ at x = 0 and $T_{s,2}$ at x = L. For constant thermal conductivity, the governing differential equation is

$$\frac{d^2T}{dx^2}+\frac{g}{k}=0 \tag{4.12}$$

with the boundary conditions of

$$x=0, h_1(T-T_{\infty,1})-k\frac{dT}{dx}=0 \tag{4.13}$$

$$x=L, h_2(T-T_{\infty,2})-k\frac{dT}{dx}=0 \tag{4.14}$$

The solution of Eq. 4.12 with the specified boundary condition is developed in example 4.8.

Example 4.8

```
> restart;
> Eq1:=diff(T(x),x,x)+g/k=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\mathrm{T}(x)\right)+\frac{g}{k}=0$$

```
> Eq2:=dsolve(Eq1,T(x));
```

$$Eq2 := \mathrm{T}(x) = -\frac{g\,x^2}{2\,k}+\_C1\,x+\_C2$$

```
> Eq3:=diff(%,x);
```

$$Eq3 := \frac{d}{dx}\mathrm{T}(x) = -\frac{g\,x}{k}+\_C1$$

```
> bc1:=h[1]*(subs(x=0,rhs(Eq2))-T[infinity,1])-k*subs(x=0,
rhs(Eq3));
```

$$bc1 := h_1\,(\_C2-T_{\infty,\,1})-k\,\_C1$$

```
> bc2:=h[2]*(subs(x=L,rhs(Eq2))-T[infinity,2])+k*subs(x=L,
rhs(Eq3));
```

$$bc2 := h_2\left(-\frac{g\,L^2}{2\,k}+\_C1\,L+\_C2-T_{\infty,\,2}\right)+k\left(-\frac{g\,L}{k}+\_C1\right)$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \left\{ \_C1 = \frac{1}{2}\frac{h_1(h_2 g L^2 + 2 h_2 T_{\infty,2} k + 2 k g L - 2 T_{\infty,1} h_2 k)}{k(h_2 k + h_2 L h_1 + k h_1)}, \right.$$

$$\left. \_C2 = \frac{1}{2}\frac{h_2 g L^2 + 2 h_2 L h_1 T_{\infty,1} + 2 h_2 T_{\infty,2} k + 2 k g L + 2 k h_1 T_{\infty,1}}{h_2 k + h_2 L h_1 + k h_1} \right\}$$

```
> assign(const):
```

```
> Eq4:=Eq2;
```
(Temperature distribution in the wall)

$$Eq4 := \mathrm{T}(x) = -\frac{g x^2}{2k} + \frac{1}{2}\frac{h_1(h_2 g L^2 + 2 h_2 T_{\infty,2} k + 2 k g L - 2 T_{\infty,1} h_2 k) x}{k(h_2 k + h_2 L h_1 + k h_1)}$$

$$+ \frac{1}{2}\frac{h_2 g L^2 + 2 h_2 L h_1 T_{\infty,1} + 2 h_2 T_{\infty,2} k + 2 k g L + 2 k h_1 T_{\infty,1}}{h_2 k + h_2 L h_1 + k h_1}$$

```
> T[s,1]:=subs(x=0,rhs(Eq4));
```
(Temperature on the left face)

$$T_{s,1} := \frac{1}{2}\frac{h_2 g L^2 + 2 h_2 L h_1 T_{\infty,1} + 2 h_2 T_{\infty,2} k + 2 k g L + 2 k h_1 T_{\infty,1}}{h_2 k + h_2 L h_1 + k h_1}$$

```
> T[s,2]:=subs(x=L,rhs(Eq4));
```
(Temperature on the right face)

$$T_{s,2} := -\frac{g L^2}{2k} + \frac{1}{2}\frac{h_1(h_2 g L^2 + 2 h_2 T_{\infty,2} k + 2 k g L - 2 T_{\infty,1} h_2 k) L}{k(h_2 k + h_2 L h_1 + k h_1)}$$

$$+ \frac{1}{2}\frac{h_2 g L^2 + 2 h_2 L h_1 T_{\infty,1} + 2 h_2 T_{\infty,2} k + 2 k g L + 2 k h_1 T_{\infty,1}}{h_2 k + h_2 L h_1 + k h_1}$$

```
> Heat_flux_left_face:=simplify(h[1]*(T[s,1]-
T[infinity,1]));
```

$$Heat\_flux\_left\_face := \frac{1}{2}\frac{h_1(h_2 g L^2 + 2 h_2 T_{\infty,2} k + 2 k g L - 2 T_{\infty,1} h_2 k)}{h_2 k + h_2 L h_1 + k h_1}$$

```
> Heat_flux_right_face:=simplify(h[2]*(T[s,2]-
T[infinity,2]));
```

$$Heat\_flux\_right\_face := \frac{1}{2}\frac{h_2 (g L^2 h_1 + 2 k g L + 2 k h_1 T_{\infty,1} - 2 T_{\infty,2} k h_1)}{h_2 k + h_2 L h_1 + k h_1}$$

```
> Total_heat_flux:=Heat_flux_left_face+Heat_flux_right_face;
```

$$Total\_heat\_flux := \frac{1}{2}\frac{h_1 (h_2 g L^2 + 2 h_2 T_{\infty,2} k + 2 k g L - 2 T_{\infty,1} h_2 k)}{h_2 k + h_2 L h_1 + k h_1} + \frac{1}{2}\frac{h_2 (g L^2 h_1 + 2 k g L + 2 k h_1 T_{\infty,1} - 2 T_{\infty,2} k h_1)}{h_2 k + h_2 L h_1 + k h_1}$$

```
> simplify(%);
```

$$g L$$

As a numerical example, let us plot the temperature distribution in the wall and calculate the surface temperatures, the location of the maximum temperature, and the maximum temperature. The following data is assumed: g = 0.3 MW/m^3, L= 0.1 m, h_1 = 400 W/m^2-K, $T_{\infty,1}$ = 20 °C, h_2= 200 W/ m^2-K, $T_{\infty,2}$ = 15 °C.

```
> g:=300000; L:=0.1; k:=25; h[1]:=400; T[infinity,
1]:=20;h[2]:=200;T[infinity,2]:=15;
```

$$g := 300000$$
$$L := 0.1$$
$$k := 25$$
$$h_1 := 400$$
$$T_{\infty,1} := 20$$
$$h_2 := 200$$
$$T_{\infty,2} := 15$$

```
> Temp_distribution:=simplify(rhs(Eq4));
```

$$Temp\_distribution := -6000.\, x^2 + 713.0434785\, x + 64.56521740$$

```
> plot(Temp_distribution,x=0..0.1,color=black,labels=["x
(m)", "T (°C)"]);
```

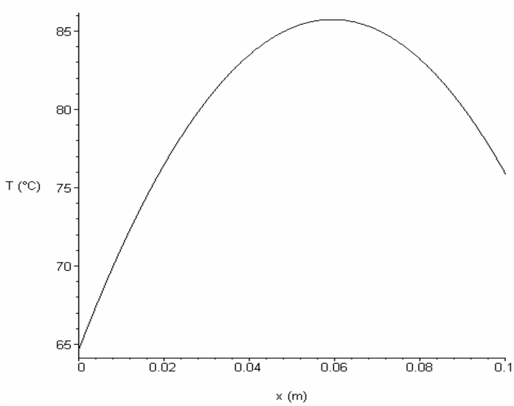


Fig 4.14: Temperature distribution in a plane wall with uniform heat generation

```
> T[s,1];
```

$$64.56521740$$

```
> T[s,2];
```

$$75.86956525$$

```
> solve(rhs(Eq3)=0,x); The location of Tmax in the wall
```

$$0.05942028988$$

```
> T[m]:=subs(x=%,Temp_distribution);
```

$$T_m := 85.74984250$$

Note that the temperature distribution in the wall (Fig 4.14) is asymmetric about the centerline because of the asymmetrical cooling on the two sides of the wall. The maximum temperature of 85.7°C occurs at x = 0.0594 m which is to the right of the centerline. This is because of the relatively weaker convection on the right face of the wall.

4.7.2 Composite Cylindrical Shell

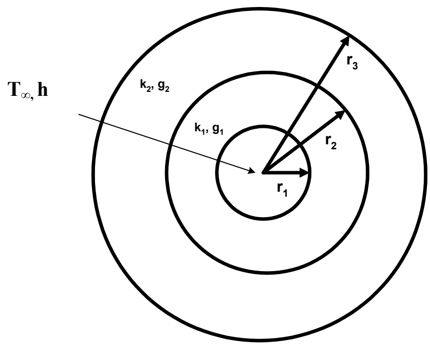


Fig: 4.15: Composite cylindrical shell with heat generation

A composite cylindrical shell is composed of two materials as shown in Fig 4.15. The inner shell has a thermal conductivity k_1 and is generating heat at the rate of g_1 (W/m^3). The outer shell has a thermal conductivity k_2 and is generating heat at the rate of g_2 (W/m^3). The surface at r_1 is cooled by a coolant at temperature T_∞ with a convective heat transfer coefficient of h. The outer surface at r_3 is insulated.

The governing equations are

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_1}{dr}\right)+\frac{g_1}{k_1}=0 \tag{4.15}$$

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dT_2}{dr}\right)+\frac{g_2}{k_2}=0 \tag{4.16}$$

with boundary conditions of

$$r=r_1, h(T_1-T_\infty)-k_1\frac{dT_1}{dr}=0 \tag{4.17}$$

$$r=r_2 \qquad T_1=T_2 \qquad k_1\frac{dT_1}{dr}=k_2\frac{dT_2}{dr} \tag{4.18}$$

$$r=r_3 \qquad \frac{dT_2}{dr}=0 \tag{4.19}$$

Example 4.9

```
> restart;
> Eq1:=(1/r)*diff(r*diff(T[1](r),r),r)+g[1]/k[1]=;
```

$$Eq1 := \frac{\left(\frac{d}{dr}T_1(r)\right)+r\left(\frac{d^2}{dr^2}T_1(r)\right)}{r}+\frac{g_1}{k_1}=0$$

```
> Eq2:=dsolve(Eq1,T[1](r));
```

$$Eq2 := T_1(r) = -\frac{1}{4}\frac{g_1 r^2}{k_1} + \_C1 \ln(r) + \_C2$$

```
> Eq3:=(1/r)*diff(r*diff(T[2](r),r),r)+g[2]/k[2]=;
```

$$Eq3 := \frac{\left(\frac{d}{dr}T_2(r)\right)+r\left(\frac{d^2}{dr^2}T_2(r)\right)}{r}+\frac{g_2}{k_2}=0$$

```
> Eq4:=dsolve(Eq3,T[2](r));
```

$$Eq4 := T_2(r) = -\frac{1}{4}\frac{g_2 r^2}{k_2} + \_C1 \ln(r) + \_C2$$

```
> Eq5:=subs(_C1=_C3,_C2=_C4,Eq4);
```

$$Eq5 := T_2(r) = -\frac{1}{4}\frac{g_2 r^2}{k_2} + \_C3 \ln(r) + \_C4$$

```
> bc1:=h*(subs(r=r[1],rhs(Eq2))-T[infinity])-k[1]*
subs(r=r[1],diff(rhs(Eq2),r));
```

$$bc1 := h\left(-\frac{1}{4}\frac{g_1 r_1^{\,2}}{k_1} + \_C1 \ln(r_1) + \_C2 - T_\infty\right) - k_1\left(-\frac{1}{2}\frac{r_1 g_1}{k_1} + \frac{\_C1}{r_1}\right)$$

```
> bc2:=subs(r=r[2],rhs(Eq2))-subs(r=r[2],rhs(Eq5));
```

$$bc2 := -\frac{1}{4}\frac{g_1 r_2^2}{k_1} + \_C1 \ln(r_2) + \_C2 + \frac{1}{4}\frac{g_2 r_2^2}{k_2} - \_C3 \ln(r_2) - \_C4$$

```
> bc3:=subs(r=r[3],diff(rhs(Eq5),r));
```

$$bc3 := -\frac{1}{2}\frac{r_3 g_2}{k_2} + \frac{\_C3}{r_3}$$

```
> bc4:=k[1]*subs(r=r[2],diff(rhs(Eq2),r))-k[2]*subs(r=r[2],
diff(rhs(Eq5),r));
```

$$bc4 := k_1\left(-\frac{1}{2}\frac{r_2 g_1}{k_1} + \frac{\_C1}{r_2}\right) - k_2\left(-\frac{1}{2}\frac{r_2 g_2}{k_2} + \frac{\_C3}{r_2}\right)$$

```
> const:=solve({bc1=0,bc2=0,bc3=0,bc4=0},{_C1,_C2,_C3,_C4});
```

$$const := \left\{\_C3 = \frac{1}{2}\frac{r_3^2 g_2}{k_2},\ \_C1 = -\frac{1}{2}\frac{-r_2^2 g_1 + r_2^2 g_2 - r_3^2 g_2}{k_1},\ \_C4 = -\frac{1}{4}(r_2^2 g_1 k_2 h r_1 - 2\ln(r_2) k_2 r_2^2 g_1 h r_1 + 2\ln(r_2) k_2 r_2^2 g_2 h r_1 - 2\ln(r_2) k_2 r_3^2 g_2 h r_1 - k_2 h r_1^3 g_1 + 2 k_2 h r_1 \ln(r_1) r_2^2 g_1 - 2 k_2 h r_1 \ln(r_1) r_2^2 g_2 + 2 k_2 h r_1 \ln(r_1) r_3^2 g_2 + 2 k_2 r_2^2 g_2 k_1 - 4 k_2 h r_1 T_\infty k_1 + 2 k_2 k_1 g_1 r_1^2 - 2 k_2 k_1 r_2^2 g_1 - 2 k_2 k_1 r_3^2 g_2 - r_2^2 g_2 k_1 h r_1 + 2 r_3^2 g_2 \ln(r_2) k_1 h r_1) \big/ (k_1 k_2 h r_1),\ \_C2 = \frac{1}{4}(h r_1^3 g_1 - 2 h r_1 \ln(r_1) r_2^2 g_1 + 2 h r_1 \ln(r_1) r_2^2 g_2 - 2 h r_1 \ln(r_1) r_3^2 g_2 - 2 r_2^2 g_2 k_1 + 4 h r_1 T_\infty k_1 - 2 k_1 g_1 r_1^2 + 2 k_1 r_2^2 g_1 + 2 k_1 r_3^2 g_2) \big/ (h r_1 k_1)\right\}$$

```
> assign(const):
```

```
> Eq6:=simplify(Eq2);
```

(Temperature Distribution in the inner shell)

$$Eq6 := T_1(r) = -\frac{1}{4}(g_1 r^2 h r_1 - 2\ln(r) h r_1 r_2^2 g_1 + 2\ln(r) h r_1 r_2^2 g_2 - 2\ln(r) h r_1 r_3^2 g_2 - h r_1^3 g_1 + 2 h r_1 \ln(r_1) r_2^2 g_1 - 2 h r_1 \ln(r_1) r_2^2 g_2 + 2 h r_1 \ln(r_1) r_3^2 g_2 + 2 r_2^2 g_2 k_1 - 4 h r_1 T_\infty k_1 + 2 k_1 g_1 r_1^2 - 2 k_1 r_2^2 g_1 - 2 k_1 r_3^2 g_2) / (h r_1 k_1)$$

```
> Eq7:=simplify(Eq5);  Temperature distribution in the outer shell
```

$$Eq7 := T_2(r) = \frac{1}{4}(-g_2 r^2 k_1 h r_1 + 2 r_3^2 g_2 \ln(r) k_1 h r_1 - r_2^2 g_1 k_2 h r_1 + 2\ln(r_2) k_2 r_2^2 g_1 h r_1 - 2\ln(r_2) k_2 r_2^2 g_2 h r_1 + 2\ln(r_2) k_2 r_3^2 g_2 h r_1 + k_2 h r_1^3 g_1 - 2 k_2 h r_1 \ln(r_1) r_2^2 g_1 + 2 k_2 h r_1 \ln(r_1) r_2^2 g_2 - 2 k_2 h r_1 \ln(r_1) r_3^2 g_2 - 2 k_2 r_2^2 g_2 k_1 + 4 k_2 h r_1 T_\infty k_1 - 2 k_2 k_1 g_1 r_1^2 + 2 k_2 k_1 r_2^2 g_1 + 2 k_2 k_1 r_3^2 g_2 + r_2^2 g_2 k_1 h r_1 - 2 r_3^2 g_2 \ln(r_2) k_1 h r_1) / (k_1 k_2 h r_1)$$

In the case of fixed inside surface temperature, T_i can be derived by allowing h to approach infinity and replacing T_∞ with T_i as shown below. The temperature distributions in this case are represented by Eq.(8) for the inner shell and Eq. (9) for the outer shell.

```
> limit(rhs(Eq6),h=infinity):Eq8:=simplify(subs(T[infinity]=T[i],%));
```

$$Eq8 := -\frac{1}{4}(g_1 r^2 - 2\ln(r) r_2^2 g_1 + 2\ln(r) r_2^2 g_2 - 2 r_3^2 g_2 \ln(r) - g_1 r_1^2 + 2\ln(r_1) r_2^2 g_1 - 2\ln(r_1) r_2^2 g_2 + 2\ln(r_1) r_3^2 g_2 - 4 T_i k_1) / k_1$$

```
> limit(rhs(Eq7),h=infinity):Eq9:=subs(T[infinity]=T[i],%);
```

(*Maple* return on following page)

$$Eq9 := \frac{1}{4}\left(-g_2\, r^2\, k_1 + 2\, r_3^{\ 2}\, g_2 \ln(r)\, k_1 - r_2^{\ 2}\, g_1\, k_2 + 2\ln(r_2)\, k_2\, r_2^{\ 2}\, g_1 - 2\ln(r_2)\, k_2\, r_2^{\ 2}\, g_2 + 2\ln(r_2)\, k_2\, r_3^{\ 2}\, g_2 + g_1\, r_1^{\ 2}\, k_2 - 2\ln(r_1)\, r_2^{\ 2}\, g_1\, k_2 + 2\ln(r_1)\, r_2^{\ 2}\, g_2\, k_2 - 2\ln(r_1)\, r_3^{\ 2}\, g_2\, k_2 + 4\, T_i\, k_1\, k_2 - 2\, r_3^{\ 2}\, g_2 \ln(r_2)\, k_1 + r_2^{\ 2}\, g_2\, k_1\right) \Big/ (k_1\, k_2)$$

```
>r[1]:=0.02;r[2]:=0.04;r[3]:=0.06;g[1]:=50000;g[2]:=200000;h
:=400;T[infinity]:=20;k[1]:=20;k[2]:=10;
```

$$r_1 := 0.02$$
$$r_2 := 0.04$$
$$r_3 := 0.06$$
$$g_1 := 50000$$
$$g_2 := 200000$$
$$h := 400$$
$$T_\infty := 20$$
$$k_1 := 20$$
$$k_2 := 10$$

```
> T_inside:=evalf(subs(r=0.02,rhs(Eq6)),5);
```

$$T\_inside := 48.751$$

```
> T_interface:=evalf(subs(r=0.03,rhs(Eq6)),5);
```

$$T\_interface := 53.303$$

```
> T_outside:=evalf(subs(r=0.04,rhs(Eq7)),5);
```

$$T\_outside := 56.32$$

```
> h*(T_inside-T[infinity])*2*Pi*r[1]:evalf(%,4);
```

$$1445.$$

```
> g[1]*Pi*(r[2]^2-r[1]^2)+g[2]*Pi*(r[3]^2-
r[2]^2):evalf(%,4);
```

$$1445.$$

The two values match, providing a check on the accuracy of the solution.

Chapter 5

Advanced One-Dimensional Steady Conduction

5.1 INTRODUCTION

The previous chapter was devoted to elementary one dimensional steady conduction problems. The analyses presented were based in assumptions such as constant thermal conductivity of the medium, uniform heat generation and pure convective heating or cooling at the surface. In many engineering situations, these assumptions are either not valid or they introduce significant errors in predicting the performance of the systems.

For example, the conducting region may be non-homogeneous and, consequently, the thermal conductivity may be location dependent. Similarly, the temperature differences involved may be significant enough to invalidate the assumption of constant conductivity. The assumption of uniform heat generation also becomes restrictive in many cases. For example, when the shield of a nuclear reactor is irradiated with gamma rays, the heat generation in the shield decays exponentially with the distance from the surface receiving the radiation, and a more appropriate heat conduction model must allow for the location dependence of heat generation. In cases where heat generation is due to electrical energy or internal chemical reaction, the heat generation becomes temperature dependent.

In some one-dimensional conduction situations, it may be necessary to consider the effect of radiation heating or cooling in addition to convection heating or cooling. This is particularly true for systems operating at high temperatures or when the boundary of the system is cooled by natural convection, when the contribution of radiation is comparable to that of natural convection. The purpose of this chapter is to investigate the problems identified in the foregoing paragraphs. In addition, two examples of optimum design of thermal systems are presented.

5.2 VARIABLE THERMAL CONDUCTIVITY

The thermal conductivity of a conducting medium may be location dependent or temperature dependent or both. Each of these cases is considered in the subsections that follow and both plane wall and hollow cylinder geometries are discussed.

5.2.1 Location Dependent Thermal Conductivity

Consider a plane wall of thickness L made of a material whose thermal conductivity dependence on x is of the form

$$k = k_o(1 + ax^2) \tag{5.1}$$

where a is a constant and k_o is the thermal conductivity at x = 0. Fig 5.1 shows the physical configuration where it is observed that the two faces of the wall are kept at fixed temperatures, T_1 and T_2. For steady one-dimensional conduction, the differential equation governing the temperature distribution in the wall is

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0 \tag{5.2}$$

with the following boundary conditions.

$$T(x = 0) = T_1 \quad and \quad T(x = L) = T_2 \tag{5.3}$$

The heat flow, q , through the wall is given by

$$q = -kA\frac{dT}{dx} \tag{5.4}$$

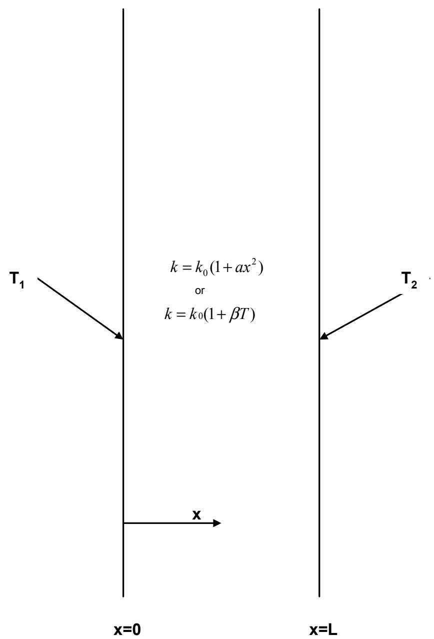


Fig. 5.1: Plane Wall showing two forms of thermal conductivity variation.

Example 5.1 : Plane Wall

```
> restart;
> k:=k0*(1+a*x^2);
```

$$k := k0\,(1 + a\,x^2)$$

```
> Eq1:=diff(k*diff(T(x),x),x)=0;
```

$$Eq1 := 2\,k0\,a\,x\left(\frac{d}{dx}\mathrm{T}(x)\right) + k0\,(1 + a\,x^2)\left(\frac{d^2}{dx^2}\mathrm{T}(x)\right) = 0$$

```
> Eq2:=dsolve({Eq1,T(0)=T[1],T(L)=T[2]},T(x));
```

Temperature Distribution

$$Eq2 := \mathrm{T}(x) = T_1 - \frac{\arctan(\sqrt{a}\,x)\,(T_1 - T_2)}{\arctan(\sqrt{a}\,L)}$$

```
> Eq3:=-k*A*diff(rhs(Eq2),x);
```

Heat transfer rate at any location

$$Eq3 := \frac{k0\,A\,\sqrt{a}\,(T_1 - T_2)}{\arctan(\sqrt{a}\,L)}$$

For a=0 (constant thermal conductivity, Eqs. 2 and 3 reduce to the standard expressions for the temperature profile and heat transfer rate for a plane wall with constant thermal conductivity as shown below.

```
> Eq4:=limit(rhs(Eq2),a=0);
```

Temperature distribution for constant k

$$Eq4 := \frac{T_1\,L - x\,T_1 + x\,T_2}{L}$$

```
> Eq5:=limit(Eq3,a=0);
```

Heat transfer rate for constant k

$$Eq5 := \frac{k0\,A\,T_1 - k0\,A\,T_2}{L}$$

Next, consider a hollow cylinder with inside radius, r_1 and outside radius r_2 fabricated of a material whose thermal conductivity is a linear function of the radial coordinate, that is,

$$k = a(1 + br) \tag{5.5}$$

where a and b are constants.

The temperatures on the inside and outside surfaces are T_1 and T_2 respectively as shown in Fig 5.2.

For steady state conduction, the governing differential equation together with the boundary conditions is

$$\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0 \tag{5.6}$$

$$T(r=r_1)=T_1 \tag{5.7a}$$

$$T(r=r_2)=T_2 \tag{5.7b}$$

The heat flow, q, is obtained by applying the Fourier equation

$$q=-k(2\pi rL)\frac{dT}{dr} \tag{5.8}$$

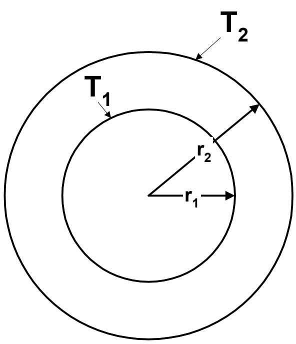


Fig 5.2: Hollow cylinder with k= a(1+b r)

Example 5.2

```
> restart;
> k:=a*(1+b*r);
```

$$k := a\,(1+b\,r)$$

```
> Eq1:=diff(k*r*diff(T(r),r),r)=0;
```

$$Eq1 := a\,b\,r\left(\frac{d}{dr}\mathrm{T}(r)\right)+a\,(1+b\,r)\left(\frac{d}{dr}\mathrm{T}(r)\right)+a\,(1+b\,r)\,r\left(\frac{d^2}{dr^2}\mathrm{T}(r)\right)=0$$

```
> Eq2:=dsolve(Eq1,T(r));
```

$$Eq2 := \mathrm{T}(r) = \_C1 + (-\ln(1+b\,r)+\ln(r))\,\_C2$$

```
> Eq3:=diff(Eq2,r);
```

$$Eq3 := \frac{d}{dr}\mathrm{T}(r) = \left(-\frac{b}{1+b\,r}+\frac{1}{r}\right)\_C2$$

```
> bc1:=subs(r=r[1],rhs(Eq2))-T[1];
```

$$bc1 := \_C1 + (-\ln(1+b\,r_1)+\ln(r_1))\,\_C2 - T_1$$

```
> bc2:=subs(r=r[2],rhs(Eq2))-T[2];
```

$$bc2 := \_C1 + (-\ln(1+b\,r_2)+\ln(r_2))\,\_C2 - T_2$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \{\_C2 = -\frac{T_1 - T_2}{-\ln(1+b\,r_2)+\ln(r_2)+\ln(1+b\,r_1)-\ln(r_1)},$$
$$\_C1 = -\frac{\ln(1+b\,r_2)\,T_1 - \ln(r_2)\,T_1 - T_2\ln(1+b\,r_1) + T_2\ln(r_1)}{-\ln(1+b\,r_2)+\ln(r_2)+\ln(1+b\,r_1)-\ln(r_1)}\}$$

```
> assign(const):
```

```
> Eq2;
```
(Temperature distribution)

$$\mathrm{T}(r) = -\frac{\ln(1+b\,r_2)\,T_1 - \ln(r_2)\,T_1 - T_2\ln(1+b\,r_1) + T_2\ln(r_1)}{-\ln(1+b\,r_2)+\ln(r_2)+\ln(1+b\,r_1)-\ln(r_1)}$$
$$-\frac{(-\ln(1+b\,r)+\ln(r))\,(T_1-T_2)}{-\ln(1+b\,r_2)+\ln(r_2)+\ln(1+b\,r_1)-\ln(r_1)}$$

```
> q:=simplify(-k*2*Pi*r*L*rhs(Eq3));
```
(Heat transfer rate)

$$q := \frac{2\,a\,\pi\,L\,(T_1-T_2)}{-\ln(1+b\,r_2)+\ln(r_2)+\ln(1+b\,r_1)-\ln(r_1)}$$

Observe that for b = 0 the forgoing solutions for T(r) and q reduce to the standard expressions for constant k.

5.2.2 Temperature Dependent Thermal Conductivity

Consider the configuration of example 5.1 where the thermal conductivity is a linear function of temperature of the form

$$k = k_o(1+\beta T) \tag{5.9}$$

Example 5.3 uses *Maple* to generate a solution.

Example 5.3

```
> restart;
> Eq1:=diff((1+beta*T(x))*diff(T(x),x),x)=0;
```

$$Eq1 := \beta\left(\frac{d}{dx}\mathrm{T}(x)\right)^2 + (1+\beta\,\mathrm{T}(x))\left(\frac{d^2}{dx^2}\mathrm{T}(x)\right) = 0$$

```
> Eq2:=dsolve({Eq1,T(0)=T[1],T(L)=T[2]},T(x));
```

$$Eq2 := \mathrm{T}(x) = \frac{1}{2}\,\frac{-2-2\sqrt{1-\dfrac{\beta\,(2\,T_1+T_1^{\,2}\,\beta-2\,T_2-T_2^{\,2}\,\beta)\,x}{L}+2\,\beta\left(T_1+\dfrac{1}{2}\,T_1^{\,2}\,\beta\right)}}{\beta},$$

$$\mathrm{T}(x) = \frac{1}{2}\,\frac{-2+2\sqrt{1-\dfrac{\beta\,(2\,T_1+T_1^{\,2}\,\beta-2\,T_2-T_2^{\,2}\,\beta)\,x}{L}+2\,\beta\left(T_1+\dfrac{1}{2}\,T_1^{\,2}\,\beta\right)}}{\beta}$$

We choose the second solution because T(x) must be positive

```
> Eq3:=Eq2[2];
```

(Temperature distribution in the wall)

$$Eq3 := \mathrm{T}(x) = \frac{1}{2}\,\frac{-2 + 2\sqrt{1 - \dfrac{\beta\,(2\,T_1 + T_1^{\,2}\,\beta - 2\,T_2 - T_2^{\,2}\,\beta)\,x}{L} + 2\,\beta\left(T_1 + \dfrac{1}{2}\,T_1^{\,2}\,\beta\right)}}{\beta}$$

```
> Eq4:=-A*k[0]*(1+beta*T[1])*subs(x=0,diff(rhs(Eq3),x)):
> radsimp(%):
> q:=collect(%,beta);
```

$$q := \frac{1}{2}\,\frac{A\,k_0\,(T_1^{\,2} - T_2^{\,2})\,\beta}{L} + \frac{1}{2}\,\frac{A\,k_0\,(2\,T_1 - 2\,T_2)}{L}$$

```
> Eq5:=collect(limit(rhs(Eq3),beta=0),x);
```

(Temperature distribution for constant k)

$$Eq5 := \frac{(-T_1 + T_2)\,x}{L} + T_1$$

```
> Eq6:=limit(q,beta=0);
```

(Heat transfer rate for constant k)

$$Eq6 := \frac{A\,k_0\,(T_1 - T_2)}{L}$$

```
> L:=0.2;T[1]:=200;T[2]:=50;k[0]:=25;A:=1;
```

$$L := 0.2$$
$$T_1 := 200$$
$$T_2 := 50$$
$$k_0 := 25$$
$$A := 1$$

```
> q;
```

$$0.2343750000\ 10^{7}\ \beta + 18750.00000$$

```
> q:=unapply(%,beta);
```

$$q := \beta \to 0.2343750000\ 10^{7}\ \beta + 18750.00000$$

```
> evalf(q(0),5);
```

$$18750.$$

```
> evalf(q(0.001),5);
```

$$21094.$$

```
> evalf(q(-0.001,5));
```

$$16406.25000$$

Note that for β>0 the average thermal conductivity is higher than in the case of constant thermal conductivity, hence the heat flow is increased. The reverse is true when β<0.

The hollow cylinder of Fig 5.2 is considered next with thermal conductivity varying linearly with T in accordance with Eq. (5.9).

Example 5.4

```
> restart;
```

```
> Eq1:=diff((1+beta*T(r))*r*diff(T(r),r),r)=0;
```

$$Eq1 := \beta \left(\frac{d}{dr}\,\mathrm{T}(r)\right)^{2} r + (1+\beta\,\mathrm{T}(r))\left(\frac{d}{dr}\,\mathrm{T}(r)\right) + (1+\beta\,\mathrm{T}(r))\,r\left(\frac{d^{2}}{dr^{2}}\,\mathrm{T}(r)\right) = 0$$

```
> dsolve(Eq1,T(r));
```

$$\mathrm{T}(r) = -\frac{1}{\beta},\ \mathrm{T}(r) = \frac{1}{2}\,\frac{-2+2\sqrt{1+2\,\beta\,\_C1\,\ln(r)+2\,\beta\,\_C2}}{\beta},$$
$$\mathrm{T}(r) = \frac{1}{2}\,\frac{-2-2\sqrt{1+2\,\beta\,\_C1\,\ln(r)+2\,\beta\,\_C2}}{\beta}$$

Because T(r) must be a positive quantity, the second solution of the set returned by *Maple* is the correct one. We may also solve the problem numerically as shown next. For the purpose of numerical solution, we assume β= 0.001, T_1= 100°C, T_2 = 50°C, r_1= 0.1 m, r_2 = 0.15. The results for the temperature distribution are generated using a radial spacing of 0.01 m.

```
> beta:=0.001;T1:=100;T2:=50;r1:=0.1;r2:=0.15;
```

$$\beta := 0.001$$
$$T1 := 100$$
$$T2 := 50$$
$$r1 := 0.1$$
$$r2 := 0.15$$

```
> Eq3:=Eq1;
```

$Eq3 :=$

$$0.001 \left(\frac{d}{dr}\mathrm{T}(r)\right)^2 r + (1 + 0.001\ \mathrm{T}(r))\left(\frac{d}{dr}\mathrm{T}(r)\right) + (1 + 0.001\ \mathrm{T}(r))\, r \left(\frac{d^2}{dr^2}\mathrm{T}(r)\right) = 0$$

```
> guesses:=[-800,-000,-1200,-1400];
```

$$guesses := [-800, 0, -1200, -1400]$$

```
> approx:=proc(guess)
> dsolve({Eq3,T(0.1)=100,D(T)(0.1)=guess},T(r),numeric);
> end:
> Eq4:=map(approx,guesses);
```

$Eq4 :=$ [**proc** (x_rkf45) ... **end proc** , **proc** (x_rkf45) ... **end proc** ,
proc (x_rkf45) ... **end proc** , **proc** (x_rkf45) ... **end proc**]

```
> for n from 1 to 4 do
> print(n,Eq4[n](0.15));
> od;
```

$$1, \left[r = 0.15, \mathrm{T}(r) = 67.0699093432479572 , \frac{d}{dr}\mathrm{T}(r) = -549.791910394582600 \right]$$

$$2, \left[r = 0.15, \mathrm{T}(r) = 100., \frac{d}{dr}\mathrm{T}(r) = 0. \right]$$

$$3, \left[r = 0.15, \mathrm{T}(r) = 50.2177237832992702 , \frac{d}{dr}\mathrm{T}(r) = -837.921508010620300 \right]$$

$$4, \left[r = 0.15, \mathrm{T}(r) = 41.6893933010125936 \,, \frac{d}{dr}\mathrm{T}(r) = -985.578530760650892 \right]$$

```
> x:=[50.218,41.689];
```

$$x := [50.218, 41.689]$$

```
> y:=[-1200,-1400];
```

$$y := [-1200, -1400]$$

```
> interp(x,y,z);
```

$$23.44940790\ z - 2377.582366$$

```
> final_guess:=subs(z=50,%);
```

$$final\_guess := -1205.111971$$

```
> Eq5:=dsolve({Eq3,T(0.1)=100,D(T)(0.1)=final_guess},T(r),
numeric);
```

$$Eq5 := \mathbf{proc}\,(x\_rkf45)\ \ldots\ \mathbf{end\ proc}$$

```
> Eq5(0.15);
```

$$\left[r = 0.15, \mathrm{T}(r) = 50.0006035450153661 \,, \frac{d}{dr}\mathrm{T}(r) = -841.665039279364919 \right]$$

```
> print(Eq5(0.1),Eq5(0.11),Eq5(0.12),Eq5(0.13),Eq5(0.14),
Eq5(0.15));
```

$$\left[r = 0.1, \mathrm{T}(r) = 100., \frac{d}{dr}\mathrm{T}(r) = -1205.1119710000 \right],$$
$$\left[r = 0.11, \mathrm{T}(r) = 88.4534705270080224 \,, \frac{d}{dr}\mathrm{T}(r) = -1107.17734522065803 \right],$$
$$\left[r = 0.12, \mathrm{T}(r) = 77.8042852068995502 \,, \frac{d}{dr}\mathrm{T}(r) = -1024.94114728022669 \right],$$
$$\left[r = 0.13, \mathrm{T}(r) = 67.9142428584426768 \,, \frac{d}{dr}\mathrm{T}(r) = -954.861291204407280 \right],$$
$$\left[r = 0.14, \mathrm{T}(r) = 58.6750894066550118 \,, \frac{d}{dr}\mathrm{T}(r) = -894.394711026189839 \right],$$
$$\left[r = 0.15, \mathrm{T}(r) = 50.0006035450153661 \,, \frac{d}{dr}\mathrm{T}(r) = -841.665039279364919 \right]$$

5.3 NONUNIFORM HEAT GENERATION

As noted in the introduction, the volumetric heat generation in a conducting medium can depend on location or temperature or both. First, the case of location dependent heat generation is considered. This is followed by analyses for temperature dependent heat generation.

5.3.1 Location Dependent Heat Generation

Consider the plane wall of Fig. 5.3. Two forms of location dependent heat generation may be considered

$$g = g_o\left(1 - \frac{x}{L}\right) \tag{5.10}$$

and

$$g = a(1-b)g_o e^{-ax} \tag{5.11}$$

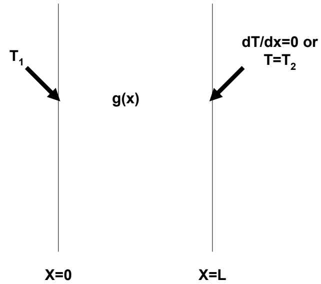


Fig. 5.3: Plane wall with location dependent heat generation

For the heat generation of the form of Eq. (5.10), we assume the boundary conditions to be

$$T(x=0) = T_1 \quad and \quad \left.\frac{dT}{dx}\right|_{x=L} = 0 \quad insulated \tag{5.12}$$

For the heat generation of the form of Eq. (5.11), the boundary conditions are assumed to be

$$T(x=0) = T_1, T(x=L) = T_2 \tag{5.13}$$

In Example 5.5, the temperature distribution is developed for the first form of heat generation. Example 5.6 presents the solution for the second form of heat generation. Example 5.6 also determines the location of the maximum temperature and an expression for the maximum temperature for the case of b = 0.

Example 5.5 Plane Wall

```
> restart:
> Eq1:=diff(T(x),x,x)+g[0]*(1-x/L)/k=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\mathrm{T}(x)\right)+\frac{g_0\left(1-\frac{x}{L}\right)}{k}=0$$

```
> Eq2:=dsolve({Eq1,T(0)=T[1],D(T)(L)=0},T(x));
```

Temperature distribution

$$Eq2 := \mathrm{T}(x)=\frac{g_0\left(-\frac{1}{2}L\,x^2+\frac{1}{6}x^3\right)}{L\,k}+\frac{1}{2}\frac{g_0\,L\,x}{k}+T_1$$

Example 5.6 Plane Wall

```
> restart;
> Eq1:=diff(T(x),x,x)+a*(1-b)*g[0]*exp(-a*x)/k=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\mathrm{T}(x)\right)+\frac{a\,(1-b)\,g_0\,\mathrm{e}^{(-a\,x)}}{k}=0$$

```
> Eq2:=dsolve({Eq1,T(0)=T[1],T(L)=T[2]},T(x));
```

$$Eq2 := \mathrm{T}(x)=\frac{(-1+b)\,g_0\,\mathrm{e}^{(-a\,x)}}{k\,a}$$
$$-\frac{(-g_0\,\mathrm{e}^{(-a\,L)}+g_0\,\mathrm{e}^{(-a\,L)}\,b+g_0-g_0\,b+T_1\,k\,a-T_2\,k\,a)\,x}{L\,k\,a}+\frac{g_0-g_0\,b+T_1\,k\,a}{k\,a}$$

```
> Eq3:=collect(Eq2,exp);
```

Temperature Distribution

$$Eq3 := \mathrm{T}(x) = -\frac{(-g_0 + g_0\, b)\, x\, \mathrm{e}^{(-a\,L)}}{L\,k\,a} + \frac{(-1+b)\, g_0\, \mathrm{e}^{(-a\,x)}}{k\,a} - \frac{(T_1\,k\,a - T_2\,k\,a + g_0 - g_0\,b)\,x}{L\,k\,a} + \frac{g_0 - g_0\,b + T_1\,k\,a}{k\,a}$$

```
> Eq4:=subs(b=0,Eq3);
```

Temperature Distribution for b=0

$$Eq4 := \mathrm{T}(x) = \frac{g_0\, x\, \mathrm{e}^{(-a\,L)}}{L\,k\,a} - \frac{g_0\, \mathrm{e}^{(-a\,x)}}{k\,a} - \frac{(T_1\,k\,a - T_2\,k\,a + g_0)\,x}{L\,k\,a} + \frac{g_0 + T_1\,k\,a}{k\,a}$$

```
> Eq5:=combine(diff(Eq4,x),exp);
```

$$Eq5 := \frac{d}{dx}\mathrm{T}(x) = \frac{g_0\, \mathrm{e}^{(-a\,L)}}{L\,k\,a} + \frac{g_0\, \mathrm{e}^{(-a\,x)}}{k} - \frac{T_1\,k\,a - T_2\,k\,a + g_0}{L\,k\,a}$$

```
> x[max]:=collect(solve(rhs(Eq5)=0,x),exp);
```

Location at which Tmax occurs

$$x_{max} := -\frac{\ln\left(-\dfrac{g_0\, \mathrm{e}^{(-a\,L)} - T_1\,k\,a + T_2\,k\,a - g_0}{g_0\,L\,a}\right)}{a}$$

```
> T[max]:=subs(x=x[max],rhs(Eq4));
```

Maximum Temperature

$$T_{max} := -\frac{g_0 \ln\left(-\dfrac{g_0\, \mathrm{e}^{(-a\,L)} - T_1\,k\,a + T_2\,k\,a - g_0}{g_0\,L\,a}\right) \mathrm{e}^{(-a\,L)}}{L\,k\,a^2} - \frac{g_0\, \mathrm{e}^{\ln\left(-\frac{g_0\, \mathrm{e}^{(-a\,L)} - T_1\,k\,a + T_2\,k\,a - g_0}{g_0\,L\,a}\right)}}{k\,a}$$

$$+\frac{(T_1\,k\,a - T_2\,k\,a + g_0)\ln\left(-\frac{g_0\,\mathrm{e}^{(-a\,L)} - T_1\,k\,a + T_2\,k\,a - g_0}{g_0\,L\,a}\right)}{L\,k\,a^2} + \frac{g_0 + T_1\,k\,a}{k\,a}$$

As an example of non uniform heat generation in a cylindrical system, consider the long solid cylindrical rod of radius, R, shown in Fig 5.4.

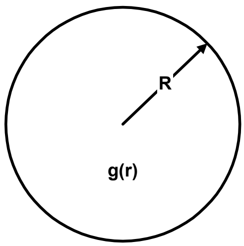


Fig. 5.4: A heat generating solid cylinder

The surface of the rod loses heat by convection to the surroundings at temperature T_∞. The interior of the rod is subjected to a distributed energy source whose strength is zero at the center where r = 0 and increases linearly with radius, that is, $g = g_0 r$. This problem is solved in Example 5.7 and expressions for the temperature distribution, the centerline temperature, the surface temperature and the heat flow from the rod are developed. The governing differential equation is

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dt}{dr}\right) + \frac{g}{k} = 0 \tag{5.14}$$

and the boundary conditions are the condition of thermal symmetry at r = 0

$$\left.\frac{dT}{dr}\right|_{r=0} = 0 \quad \textit{thermal symmetry} \tag{5.15a}$$

and at r = R

$$h(T - T_\infty) + k\frac{dT}{dr} = 0 \tag{5.15b}$$

Example 5.7 Solid cylinder

```
> restart: Eq1:=(1/r)*diff(r*diff(T(r),r),r)+g[0]*r/k=0;
```

$$Eq1 := \frac{\left(\frac{d}{dr}\mathrm{T}(r)\right) + r\left(\frac{d^2}{dr^2}\mathrm{T}(r)\right)}{r} + \frac{g_0\, r}{k} = 0$$

```
> Eq2:=dsolve(Eq1,T(r));
```

$$Eq2 := \mathrm{T}(r) = -\frac{1}{9}\frac{g_0\, r^3}{k} + \_C1\ \ln(r) + \_C2$$

```
> Eq3:=diff(Eq2,r);
```

$$Eq3 := \frac{d}{dr}\mathrm{T}(r) = -\frac{1}{3}\frac{r^2\, g_0}{k} + \frac{\_C1}{r}$$

Due to the thermal symmetry of the problem, the temperature gradient at r=0 must vanish. This can be ensured by putting _C1=0.

```
> Eq4:=subs(_C1=0,Eq2);
```

$$Eq4 := \mathrm{T}(r) = -\frac{1}{9}\frac{g_0\, r^3}{k} + \_C2$$

```
> Eq5:=subs(_C1=0,Eq3);
```

$$Eq5 := \frac{d}{dr}\mathrm{T}(r) = -\frac{1}{3}\frac{r^2\, g_0}{k}$$

```
> _C2:=solve(h*(subs(r=R,rhs(Eq4))-
T[infinity])+k*subs(r=R,rhs(Eq5))=0,_C2);
```

$$\_C2 := \frac{1}{9}\frac{h\, g_0\, R^3 + 9\, h\, T_\infty\, k + 3\, R^2\, g_0\, k}{k\, h}$$

```
> Eq6:=Eq4;
```

$$Eq6 := \mathrm{T}(r) = -\frac{1}{9}\frac{g_0\, r^3}{k} + \frac{1}{9}\frac{h\, g_0\, R^3 + 9\, h\, T_\infty\, k + 3\, R^2\, g_0\, k}{k\, h}$$

```
> T[c]:=subs(r=0,rhs(Eq6));
```

Centerline Temperature

$$T_c := \frac{1}{9}\,\frac{h\,g_0\,R^3 + 9\,h\,T_\infty\,k + 3\,R^2\,g_0\,k}{k\,h}$$

```
> T[s]:=subs(r=R,rhs(Eq6));
```

Surface Temperature

$$T_s := -\frac{1}{9}\,\frac{g_0\,R^3}{k} + \frac{1}{9}\,\frac{h\,g_0\,R^3 + 9\,h\,T_\infty\,k + 3\,R^2\,g_0\,k}{k\,h}$$

```
> q_conduction:=-k*2*Pi*R*L*subs(r=R,diff(rhs(Eq4),r));
```

Conduction Heat Transfer at the surface

$$q\_conduction := \frac{2}{3}\,\pi\,R^3\,L\,g_0$$

```
> q_convection:=simplify(h*2*Pi*R*L*(T[s]-T[infinity]));
```

Convective Heat loss from the surface

$$q\_convection := \frac{2}{3}\,\pi\,R^3\,L\,g_0$$

Note that the conduction and convection heat transfer rates match as they should.

As a numerical example, let us consider the following data and evaluate the centerline temperature, the surface temperature, the conduction heat transfer, and the convective heat transfer.

```
> R:=0.1;L:=1;g[0]:=100000;k:=5;T[infinity]:=27;h:=25;
```

$$R := 0.1$$

$$L := 1$$

$$g_0 := 100000$$

$$k := 5$$

$$T_\infty := 27$$

$$h := 25$$

```
> evalf(T[c],4);
```

$$42.56$$

```
> evalf(T[s],4);
```

40.34

```
> evalf(q_conduction,4);
```

209.5

```
> evalf(q_convection,4);
```

209.5

5.3.2 Temperature Dependent Heat Generation

This section considers one dimensional steady state conduction in which the volumetric heat generation increases linearly with temperature. Three geometries are considered:

A plane wall of thickness, 2L, (Fig 5.5)
A solid cylinder of radius, r, (Fig 5.6a)
A solid sphere of radius, r, (Fig 5.6b)

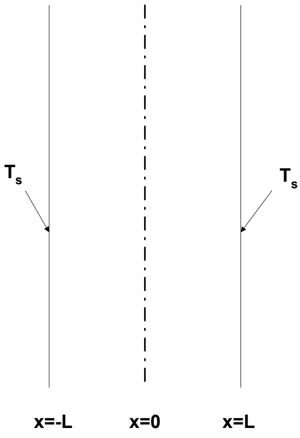


Fig. 5.5. Plane wall

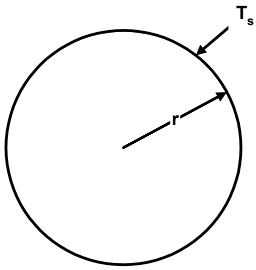


Fig. 5.6. (a) cylinder (b) sphere

The thermal conductivity for each material is k and the outside surface for each geometry is assumed to be T_s. The volumetric heat generation is expressed as

$$g = a[1 + b(T - T_s)] \tag{5.16}$$

Defining $\theta = T - T_s$, the heat conduction equations together with the boundary conditions may be summarized for the three geometries as follows.

- Plane Wall:

$$\frac{d^2\theta}{dx^2} + \frac{a(1+b\theta)}{k} = 0 \tag{5.17}$$

$$\left.\frac{d\theta}{dx}\right|_{x=0} = 0 \quad and \quad \theta(x = L) = 0 \tag{5.18}$$

- Solid Cylinder:

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \frac{a(1+b\theta)}{k} = 0 \tag{5.19}$$

$$\left.\frac{d\theta}{dr}\right|_{r=0} = 0 \quad and \quad \theta(r = R) = 0 \tag{5.20}$$

- Solid Sphere:

$$\frac{d^2\theta}{dr^2} + \frac{2}{r}\frac{d\theta}{dr} + \frac{a(1+b\theta)}{k} = 0 \tag{5.21}$$

$$\left.\frac{d\theta}{dr}\right|_{r=0} = 0 \quad and \quad \theta(r = R) = 0 \tag{5.22}$$

Examples 5.8, 5.9 and 5.10 provide solutions for the temperature distributions and the maximum temperatures (occurring at the centerline or center) for the three geometries.

Example 5.8 Plane Wall

```
> restart;
```

```
> assume(a>0,b>0,k>0);
```

The assume function allows *Maple* to simplify some expressions based on the chosen assumptions. See **assume** in the help menu for more details on the use of this command.

```
> Eq1:=diff(theta(x),x,x)+a*(1+b*theta(x))/k=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\theta(x)\right) + \frac{a\sim(1+b\sim\theta(x))}{k\sim} = 0$$

```
> Eq2:=dsolve({Eq1,D(theta)(0)=0,theta(L)=0},theta(x));
```

$$Eq2 := \theta(x) = \frac{\cos\left(\frac{\sqrt{a\sim b\sim k\sim}\, x}{k\sim}\right)}{\cos\left(\frac{\sqrt{a\sim b\sim k\sim}\, L}{k\sim}\right) b\sim} - \frac{1}{b\sim}$$

```
> Eq3:=simplify(Eq2);
```

$$Eq3 := \theta(x) = \frac{\cos\left(\frac{\sqrt{a\sim}\sqrt{b\sim}\, x}{\sqrt{k\sim}}\right) - \cos\left(\frac{\sqrt{a\sim}\sqrt{b\sim}\, L}{\sqrt{k\sim}}\right)}{\cos\left(\frac{\sqrt{a\sim}\sqrt{b\sim}\, L}{\sqrt{k\sim}}\right) b\sim}$$

```
> simplify(subs(x=0,Eq3));
```

$$\theta(0) = -\frac{-1 + \cos\left(\frac{\sqrt{a\sim}\sqrt{b\sim}\, L}{\sqrt{k\sim}}\right)}{\cos\left(\frac{\sqrt{a\sim}\sqrt{b\sim}\, L}{\sqrt{k\sim}}\right) b\sim}$$

Example 5.9 Solid Cylinder

```
> restart;
```

```
> Eq1:=diff(theta(r),r,r)+1/r*diff(theta(r),r)+a*
(1+b*theta(r))/k=0;
```

$$Eq1 := \left(\frac{d^2}{dr^2}\theta(r)\right) + \frac{\frac{d}{dr}\theta(r)}{r} + \frac{a(1+b\,\theta(r))}{k} = 0$$

```
> Eq2:=dsolve(Eq1,theta(r));
```

$$Eq2 := \theta(r) = \mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,r\right)\_C2 + \mathrm{BesselY}\left(0, \sqrt{\frac{a\,b}{k}}\,r\right)\_C1 - \frac{1}{b}$$

Because the temperature at the centerline must be finite, we put _C1=0

```
> Eq3:=subs(_C1=0,Eq2);
```

$$Eq3 := \theta(r) = \mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,r\right)\_C2 - \frac{1}{b}$$

```
> _C2:=solve(subs(r=R,rhs(Eq3))=0,_C2);
```

$$\_C2 := \frac{1}{\mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,R\right)b}$$

```
> Eq4:=Eq3;
```

$$Eq4 := \theta(r) = \frac{\mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,r\right)}{\mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,R\right)b} - \frac{1}{b}$$

```
> simplify(subs(r=0,%));
```

$$\theta(0) = -\frac{-1 + \mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,R\right)}{\mathrm{BesselJ}\left(0, \sqrt{\frac{a\,b}{k}}\,R\right)b}$$

Example 5.10 Solid Sphere

```
> restart;
```

```
> Eq1:=diff(theta(r),r,r)+2/r*diff(theta(r),r)+a*
(1+b*theta(r))/k=0;
```

$$Eq1 := \left(\frac{d^2}{dr^2}\theta(r)\right) + \frac{2\left(\frac{d}{dr}\theta(r)\right)}{r} + \frac{a\,(1+b\,\theta(r))}{k} = 0$$

```
> Eq2:=dsolve(Eq1,theta(r));
```

$$Eq2 := \theta(r) = \frac{\sin\left(\sqrt{\frac{a\,b}{k}}\,r\right)\_C2}{r} + \frac{\cos\left(\sqrt{\frac{a\,b}{k}}\,r\right)\_C1}{r} - \frac{1}{b}$$

For theta to be finite at the center(r=0) _C1 must vanish.

```
> _C1:=0;
```

$$\_C1 := 0$$

```
> Eq3:=Eq2;
```

$$Eq3 := \theta(r) = \frac{\sin\left(\sqrt{\frac{a\,b}{k}}\,r\right)\_C2}{r} - \frac{1}{b}$$

```
> _C2:=solve(subs(r=R,rhs(Eq3))=0,_C2);
```

$$\_C2 := \frac{R}{\sin\left(\sqrt{\frac{a\,b}{k}}\,R\right)b}$$

```
> Eq4:=radsimp(Eq3);
```

$$Eq4 := \theta(r) = \frac{\sin\left(\frac{\sqrt{a\,b\,k}\,r}{k}\right)R - r\sin\left(\frac{\sqrt{a\,b\,k}\,R}{k}\right)}{r\sin\left(\frac{\sqrt{a\,b\,k}\,R}{k}\right)b}$$

```
> theta(0):=simplify(limit(rhs(%),r=0));
```

$$\theta(0) := \frac{\sqrt{a\,b\,k}\,R - \sin\left(\frac{\sqrt{a\,b\,k}\,R}{k}\right)k}{k\sin\left(\frac{\sqrt{a\,b\,k}\,R}{k}\right)b}$$

5.4 COMBINED RADIATIVE-CONDUCTIVE COOLING OF SOLIDS WITH UNIFORM INTERNAL HEAT GENERATION

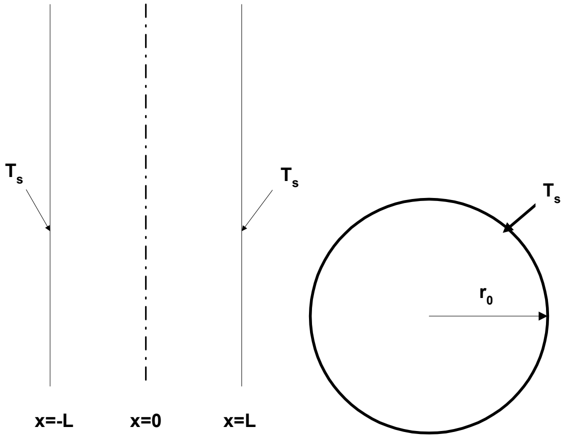


Fig. 5.7: Configurations for the study of combined radiation-convection cooling (a) the plane wall and (b) the solid cylinder and sphere

Consider one-dimensional steady conduction in a plane wall of thickness, 2L, a solid cylinder of radius r_o, and a solid sphere of radius r_o in the presence of uniform internal heat generation, g. as shown in Fig. 5.7. The thermal conductivity, k, is assumed to be constant. The outer surface in each case is cooled by radiation and convection by the environment at temperature, T_a while in all cases, the surface temperature is designated by T_s. The governing differential equations and boundary conditions are:

- Plane Wall:

$$\frac{d^2T}{dx^2}+\frac{g}{k}=0 \tag{5.23}$$

$$\left.\frac{dT}{dx}\right|_{x=0}=0 \quad and \quad T(x=L)=T_s \tag{5.24}$$

- Solid Cylinder:

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{g}{k} = 0 \tag{5.25}$$

$$\frac{dT}{dr}(r = 0) = 0, T(r = r_0) = T_s \tag{5.26}$$

- Solid Sphere:

$$\frac{d^2T}{dr^2} + \frac{2}{r}\frac{dT}{dr} + \frac{g}{k} = 0 \tag{5.27}$$

$$\frac{dT}{dr}(r = 0) = 0, T(r = r_0) = T_s \tag{5.28}$$

The surface energy balance for the plane wall at x = L gives

$$h(T_s - T_a) + \varepsilon\sigma(T_s^4 - T_a^4) + k\frac{dT}{dx} = 0 \tag{5.29}$$

and for the cylinder and sphere at $r = r_o$

$$h(T_s - T_a) + \varepsilon\sigma(T_s^4 - T_a^4) + k\frac{dT}{dr} = 0 \tag{5.30}$$

Note that for the case of the plane wall, *x* is measured from the centerline and that in all cases, *k* is the thermal conductivity of the medium, *h* is the convective heat transfer coefficient, ε is the surface emissivity, and σ is the *Stefan-Boltzmann* constant. Now we define several dimensionless quantities:

- Plane Wall:

$$X = x/L, \theta = T/T_a, G = gL^2/kT_a, N_1 = \varepsilon\sigma T_a{}^3 L/k, N_2 = hL/k$$

- For the Cylinder and sphere:

$$R = r/r_0, \theta = T/T_a, G = gr_0{}^2/kT_a, N_1 = \varepsilon\sigma T_a{}^3 r_0/k, N_2 = hr_0/k$$

With these dimensionless quantities in hand, the governing equations and boundary conditions become

- Plane Wall:

$$\frac{d^2\theta}{dx^2} + G = 0 \tag{5.31}$$

$$\left.\frac{d\theta}{dX}\right|_{X=0} = 0 \quad and \quad \theta(X=1) = \theta_s = \frac{T_s}{T_a} \tag{5.32}$$

- Solid Cylinder:

$$\frac{d^2\theta}{dR^2} + \frac{1}{R}\frac{d\theta}{dR} + G = 0 \tag{5.33}$$

$$\left.\frac{d\theta}{dR}\right|_{R=0} = 0 \quad and \quad \theta(R=1) = \theta_s = \frac{T_s}{T_a} \tag{5.34}$$

- Solid Sphere:

$$\frac{d^2\theta}{dR^2} + \frac{2}{R}\frac{d\theta}{dR} + G = 0 \tag{5.35}$$

$$\left.\frac{d\theta}{dR}\right|_{R=0} = 0 \quad and \quad \theta(R=1) = \theta_s = \frac{T_s}{T_a} \tag{5.36}$$

The surface energy balances are for the plane wall at X=1

$$N_2(\theta - 1) + N_1(\theta^4 - 1) + \frac{d\theta}{dX} = 0 \tag{5.37}$$

and for the cylinder and sphere at R=1

$$N_2(\theta - 1) + N_1(\theta^4 - 1) + \frac{d\theta}{dR} = 0 \tag{5.38}$$

Equations (5.31) to (5.36) are solved first. The unknown surface temperature is then obtained by solving the energy balances of eqs (5.37) and (5.38). Numerical values of θ_s are generated for G = 1 and a range for values for N_1 and N_2. The analyses for the plane wall, the solid cylinder and the solid sphere appear in Examples 5.11, 5.12, and 5.13, respectively.

Example 5.11 Plane Wall

```
> restart;
> Eq1:=diff(theta(X),X,X)+G=0;
```

$$Eq1 := \left(\frac{d^2}{dX^2}\theta(X)\right) + G = 0$$

```
> Eq2:=dsolve({Eq1,D(theta)(0)=0,theta(1)=theta[s]},
theta(X));
```

$$Eq2 := \theta(X) = -\frac{G\,X^2}{2} + \frac{G}{2} + \theta_s$$

```
> Eq3:=diff(Eq2,X);
```

$$Eq3 := \frac{d}{dX}\theta(X) = -G\,X$$

```
> Eq4:=subs(X=1,rhs(Eq3))+N[1]*(subs(X=1,rhs(Eq2))^4-
1)+N[2]*(subs(X=1,rhs(Eq2))-1);
```

$$Eq4 := -G + N_1\,(\theta_s^{\,4} - 1) + N_2\,(\theta_s - 1)$$

```
> Eq5:=solve(Eq4=0,theta[s]);
```

$$Eq5 := \mathrm{RootOf}(-G + N_1\,\_Z^4 - N_1 + N_2\,\_Z - N_2)$$

```
> Eq6:=[allvalues(Eq5)]:
```

(Eq5 has 4 roots that are suppressed to save space)

```
> G:=1;
```

$$G := 1$$

```
> for N[1] from 0.25 by 0.25 to 1 do
> for N[2] from 0.25 by 0.25 to 1 do
> sol:=evalf(Eq6):
> theta[s]:=sol[3]:
```

(We select the third solution because the others are negative or complex)

```
> print(N[1],N[2],theta[s]);od:od;
```

0.25, 0.25, 1.459723425
0.25, 0.50, 1.426953326
0.25, 0.75, 1.397015988
0.25, 1.00, 1.369807215
0.50, 0.25, 1.299342752
0.50, 0.50, 1.283781669
0.50, 0.75, 1.269334534
0.50, 1.00, 1.255937544
0.75, 0.25, 1.225839339
0.75, 0.50, 1.216370096
0.75, 0.75, 1.207488221
0.75, 1.00, 1.199158374
1.00, 0.25, 1.182370947
1.00, 0.50, 1.175911120
1.00, 0.75, 1.169806117
1.00, 1.00, 1.164035140

Example 5.12 Solid Cylinder

```
> restart;
```

```
> Eq1:=(1/R)*diff(R*diff(theta(R),R),R)+G=0;
```

$$Eq1 := \frac{\left(\frac{d}{dR}\theta(R)\right) + R\left(\frac{d^2}{dR^2}\theta(R)\right)}{R} + G = 0$$

```
> Eq2:=dsolve({Eq1,D(theta)(0)=0,theta(1)=theta[s]},
theta(R));
```

$$Eq2 := \theta(R) = -\frac{G\,R^2}{4} + \frac{G}{4} + \theta_s$$

```
> Eq3:=diff(Eq2,R);
```

$$Eq3 := \frac{d}{dR}\theta(R) = -\frac{G\,R}{2}$$

```
> Eq4:=subs(R=1,rhs(Eq3))+N[1]*(subs(R=1,rhs(Eq2))**4-
1)+N[2]*(subs(R=1,rhs(Eq2))-1);
```

$$Eq4 := -\frac{G}{2} + N_1\,(\theta_s^{\;4} - 1) + N_2\,(\theta_s - 1)$$

```
> Eq5:=solve(Eq4=0,theta[s]);
```

$$Eq5 := \text{RootOf}(-G + 2\,N_1\,\_Z^4 - 2\,N_1 + 2\,N_2\,\_Z - 2\,N_2)$$

```
> Eq6:=[allvalues(Eq5)]:
> G:=1;
```

$$G := 1$$

```
> for N[1] from 0.25 by 0.25 to 1 do
> for N[2] from 0.25 by 0.25 to 1 do
> sol:=evalf(Eq6):
> theta[s]:=sol[3]:
> print(N[1],N[2],theta[s]);od:od;
```

0.25, 0.25, 1.283781660
0.25, 0.50, 1.255937549
0.25, 0.75, 1.232020068
0.25, 1.00, 1.211478586
0.50, 0.25, 1.175911103
0.50, 0.50, 1.164035135
0.50, 0.75, 1.153415441
0.50, 1.00, 1.143901111
0.75, 0.25, 1.128828652
0.75, 0.50, 1.122086185
0.75, 0.75, 1.115924683

0.75, 1.00, 1.110283512
1.00, 0.25, 1.101950537
1.00, 0.50, 1.097571643
1.00, 0.75, 1.093512617
1.00, 1.00, 1.089744003

Example 5.13 Solid Sphere

```
> restart;
```

```
> Eq1:=diff(theta(R),R,R)+2/R*diff(theta(R),R)+G=;
```

$$Eq1 := \left(\frac{d^2}{dR^2}\theta(R)\right) + \frac{2\left(\frac{d}{dR}\theta(R)\right)}{R} + G = 0$$

```
> Eq2:=dsolve(Eq1,theta(R));
```

$$Eq2 := \theta(R) = -\frac{G R^2}{6} - \frac{\_C1}{R} + \_C2$$

```
> Eq3:=diff(Eq2,R);
```

$$Eq3 := \frac{d}{dR}\theta(R) = -\frac{G R}{3} + \frac{\_C1}{R^2}$$

(For the temperature gradient to be zero at the center, we set _C1=0)

```
> _C1:=0;
```

$$\_C1 := 0$$

```
> _C2:=solve(subs(R=1,rhs(Eq2))=theta[s],_C2);
```

$$\_C2 := \frac{G}{6} + \theta_s$$

```
> Eq4:=Eq2;
```

$$Eq4 := \theta(R) = -\frac{G R^2}{6} + \frac{G}{6} + \theta_s$$

```
> Eq5:=subs(R=1,rhs(Eq3))+N[1]*(subs(R=1,rhs(Eq2))^4-1)+
N[2]*(subs(R=1,rhs(Eq2))-1);
```

$$Eq5 := -\frac{G}{3} + N_1 (\theta_s{}^4 - 1) + N_2 (\theta_s - 1)$$

```
> Eq5:=solve(Eq5=0,theta[s]);
```

$$Eq5 := \mathrm{RootOf}(-G + 3\, N_1\, \_Z^4 - 3\, N_1 + 3\, N_2\, \_Z - 3\, N_2)$$

```
> Eq6:=[allvalues(Eq5)]:
> G:=1;
```

$$G := 1$$

```
> for N[1] from 0.25 by 0.25 to 1 do
> for N[2] from 0.25 by 0.25 to 1 do
> sol:=evalf(Eq6):
> theta[s]:=sol[3]:
> print(N[1],N[2],theta[s]);od:od;
```

0.25, 0.25, 1.207488225
0.25, 0.50, 1.184015404
0.25, 0.75, 1.164600236
0.25, 1.00, 1.148443782
0.50, 0.25, 1.125380762
0.50, 0.50, 1.115924693
0.50, 0.75, 1.107640802
0.50, 1.00, 1.100350325
0.75, 0.25, 1.090518886
0.75, 0.50, 1.085332126
0.75, 0.75, 1.080654512
0.75, 1.00, 1.076421856
1.00, 0.25, 1.070976812
1.00, 0.50, 1.067685365

1.00, 0.75, 1.064662510
1.00, 1.00, 1.061879323

The results for the three geometries show that as N_1 and N_2 increase, that is, as convection and radiation become stronger, the surface temperature decreases as expected.

5.5 OPTIMUM DESIGN OF THERMAL SYSTEMS

This section considers two examples of optimum design of thermal systems. The first example deals with the minimum cost design of a composite wall consisting of three different materials. The design has to meet the requirements of minimum total thermal resistance and minimum load bearing capacity. This example is solved by using the simplex package in *Maple*. The second example considers the insulation of a pipe carrying a hot fluid. The outer diameter is minimized to meet the constraint on the total annual operating cost. This is solved by reading the extrema library in *Maple*. The extrema algorithm incorporates the Lagrange multiplier method.

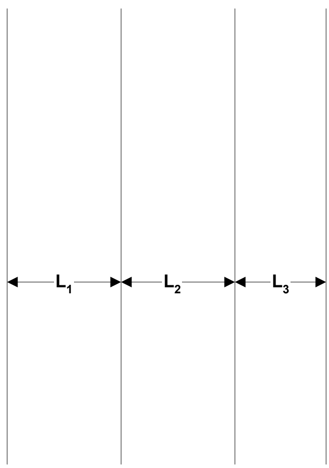


Fig. 5.8: Composite wall for Example 5.14

Example 5.14:
A composite wall consisting of three materials is shown in Fig. 5.8. The thermal resistance, load bearing capacity, and cost of the three materials, each based on one centimeter of thickness, are as shown in the following table.

Material	Thermal Resistance	Load Bearing Capacity	Cost
1	30	7	8
2	20	2	4
3	10	6	3

The total thermal resistance of the wall must be 120 or greater and the total bearing capacity must be 42 or greater. The thickness of the materials and the minimum cost are to be found.

Let L_1, L_2, and L_3 be the thicknesses in cm. The cost function and the constraints can be written as

$$\text{cost} = 8L_1 + 4L_2 + 3L_3$$
$$c_1 = 30L_1 + 20L_2 + 10L_3 \geq 120$$
$$c_2 = 7L_1 + 2L_2 + 6L_4 \geq 42$$

```
> restart;
> with(simplex):
```
Warning, the protected names maximize and minimize have been redefined and unprotected

```
> cost:=8*L[1]+4*L[2]+3*L[3];
```
$$cost := 8\,L_1 + 4\,L_2 + 3\,L_3$$

```
> c1:=30*L[1]+20*L[2]+10*L[3]>=120;
```
$$c1 := 120 \leq 30\,L_1 + 20\,L_2 + 10\,L_3$$

```
> c2:=7*L[1]+2*L[2]+6*L[3]>=42;
```
$$c2 := 42 \leq 7\,L_1 + 2\,L_2 + 6\,L_3$$

```
> minimize(cost,{c1,c2}):
> feasible({c1,c2});
```
$$true$$

```
> minimize(cost,{c1,c2},NONNEGATIVE);
```

$$\{L_1 = 0, L_2 = 3, L_3 = 6\}$$

```
> Minimum_cost:=subs(%,cost);
```

$$Minimum\_cost := 30$$

The minimum cost design requires that only materials 2 and 3 be used in the wall.

Example 5.15:

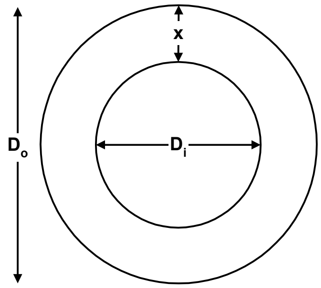


Fig. 5.9: Insulated pipe mounted in restricted space

A pipe carrying a hot fluid is to be covered with insulation and then mounted in a restricted space as shown in Fig 5.9. The choice of pipe diameter, D_i in meters, and the thickness of the insulation, x, also in meters are to be such that the outer diameter, D_o is minimized, but such that the total annual operating cost and the cost of the insulation is limited to $40,000. This cost consists of the fluid pumping loss and the cost of the heat loss through the system and is given by

$$\text{cost constraint}: \frac{8}{D_i^5} + \frac{1500}{x} = 40{,}000 \tag{5.39}$$

```
> restart;
```

```
> readlib(extrema);
```

proc (*fcn*, *cnstrnts*, *vars*, *candidates*) ... **end proc**

```
> Do:=Di+2*x;
```

$$Do := Di + 2\,x$$

```
> cost_constraint:=8/(Di^5)+1500/x-40000=0;
```

$$cost\_constraint := \frac{8}{Di^5} + \frac{1500}{x} - 40000 = 0$$

```
> extrema(Do,{cost_constraint},{Di,x},'s');
```

$$\{\max\left(\frac{4}{5}\,\mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4)\,(1 + 1250\,\mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4)^5)\right), \min\left(\frac{4}{5}\,\mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4)\,(1 + 1250\,\mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4)^5)\right)\}$$

```
> s;
```

$$\{\{Di = \mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4), x = 500\,\mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4)^6 - \frac{1}{10}\,\mathrm{RootOf}(1 - 10000\,\_Z^5 + 25000000\,\_Z^{10} - 1875\,\_Z^4)\}\}$$

```
> [allvalues(%)]:
```

```
> evalf(%);
```

$[\{\{Di = 0.1343277249, x = -0.01049536661\}\},$
$\{\{Di = 0.2314250240, x = 0.05366999915\}\}, \{$
$\{Di = 0.09557441029 + 0.2049767424\, I, x = 0.04838392337 + 0.01296930507\, I\}\},$
$\{$
$\{Di = 0.01829259049 + 0.1397705366\, I, x = -0.004615743417 - 0.01121598194\, I\}$
$\},$
$\{\{Di = -0.1340051025 + 0.1597966030\, I, x = 0.03403074382 + 0.01960759463\, I\}\}$
$, \{$
$\{Di = -0.1627382727 + 0.05123199419\, I, x = 0.01311375996 - 0.01704327504\, I\}\}$
$, \{$
$\{Di = -0.1627382727 - 0.05123199419\, I, x = 0.01311375996 + 0.01704327504\, I\}\}$
$,$
$\{\{Di = -0.1340051025 - 0.1597966030\, I, x = 0.03403074382 - 0.01960759463\, I\}\}$
$, \{$
$\{Di = 0.01829259049 - 0.1397705366\, I, x = -0.004615743417 + 0.01121598194\, I\}$
$\}, \{$
$\{Di = 0.09557441029 - 0.2049767424\, I, x = 0.04838392337 - 0.01296930507\, I\}\}]$

```
> %[2];
```

$$\{\{Di = 0.2314250240, x = 0.05366999915\}\}$$

The only solution that is physically meaningful is the last solution which gives a pipe diameter of 0.2314 meters and an insulation thickness of 0.0537 meters. This gives an outer diameter of 0.3388 meters.

Chapter 6

Extended Surfaces

6.1 INTRODUCTION

The term *extended surface* is commonly used to describe a system in which the surface area of a heated surface is increased by the attachment of fins. An extended surface accommodates energy transfer by conduction within its boundaries while its exposed surfaces transfer energy to the surroundings by convection or radiation or both. Many innovative designs of extended surfaces are used in engineering practice. Common examples are the cooling fins on electronic components, on automobile radiators or engine cylinders, on the condenser tubes of household refrigerators and on automatic control devices. A comprehensive treatment of extended surface technology has been provided by Kraus, Aziz and Welty [1].

In this chapter, it is shown how *Maple* provides an efficient tool for the analysis of a wide variety of extended surface geometries under steady operational conditions.

6.2 THE GENERAL FIN EQUATION

Consider the extended surface shown in Fig. 6.1. The general differential equation is usually derived based on the following assumptions.

- Heat conduction occurs in the longitudinal, x, direction only.
- The fin operates under steady state condition.
- The thermal conductivity, k, of the fin material is constant.
- The exposed surfaces of the fin lose heat to the surroundings at temperature T_a by convection only.

- There is no heat generation in the fin.
- The convective heat transfer coefficient is uniform over the surface of the fin.

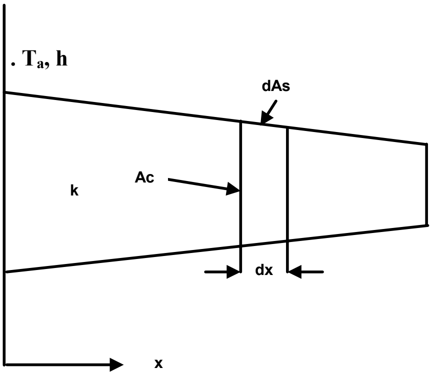


Fig. 6.1: Typical Extended Surface

Incropera and DeWitt [2] have derived the following general fin equation.

$$\frac{d}{dx}\left(kA_c\frac{dT}{dx}\right)-\frac{h}{k}\frac{dA_s}{dx}(T-T_a)=0 \qquad (6.1)$$

where A_c is the cross-sectional area at x and A_s is the surface area measured from the fin base to x.

The solution of Eq. (6.1) requires two boundary conditions and once the solution for the temperature distribution has been obtained, it can then be used to determine the heat transfer rate by applying the Fourier law at the base of the fin, that is,

$$q=-kA_c\left.\frac{dT}{dx}\right|_{x=0} \qquad (6.2)$$

6.3 STRAIGHT FINS

Four common shapes of straight fins are those of rectangular, trapezoidal, triangular, and concave parabolic profiles. The general fin differential equation of Eq. (6.1) is used to develop the specific differential equation for each geometry. Because the general fin differential equation is of second order, it requires the use of two boundary conditions for its solution. A variety of boundary conditions will be utilized in the sections to follow.

6.3.1 Rectangular Fins

For the rectangular fin, shown in Fig. 6.2, $A_c = w\ b$ where b is the width and if b is 1 m , then $A_c = w$. The surface area is $A_s = P\ x$ where P is the fin perimeter. Accordingly, $dA_c/dx = 0$ and $dA_s/dx = P$ and Eq. (6.1) reduces to

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_a) = 0 \tag{6.3}$$

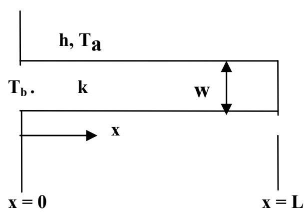


Fig. 6.2: Rectangular Fin

To simplify the form of this equation, a new dependent variable is introduced

$$\theta = T - T_a$$

allowing Eq. (6.3) to be expressed as

$$\frac{d^2\theta}{dx^2} - m^2\theta = 0 \tag{6.4}$$

where

$$m^2 = \frac{hP}{kA_c}$$

Equation (6.4) is a linear, homogeneous, differential equation with constant coefficients. To begin our analysis, we first create the differential equation of Eq. (6.4) and then obtain the general solution using the dsolve command. The general solution is differentiated with the command diff to obtain the derivative which is needed in Eq. (6.2) to evaluate the heat transfer rate and which may be needed in some of the boundary conditions. Equation (6.4) is solved using three sets of boundary conditions.

Example 6.1: Constant base temperature and insulated tip
The appropriate boundary conditions for this case are:

$$\theta(x=0)=\theta_b=T_b-T_a \tag{6.5}$$

and

$$\left.\frac{d\theta}{dx}\right|_{x=L}=0 \tag{6.6}$$

```
> restart;
> Eq1:=diff(theta(x),x,x)-m^2*theta(x)=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\theta(x)\right)-m^2\,\theta(x)=0$$

```
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x)=\_C1\,e^{(-m\,x)}+\_C2\,e^{(m\,x)}$$

```
> Eq3:=convert(Eq2,trig);
```

$$Eq3 := \theta(x)=\_C1\,(\cosh(m\,x)-\sinh(m\,x))+\_C2\,(\cosh(m\,x)+\sinh(m\,x))$$

```
> Eq4:=diff(Eq3,x);
```

$$Eq4 := \frac{d}{dx}\theta(x)=\_C1\,(\sinh(m\,x)\,m-\cosh(m\,x)\,m)+\_C2\,(\sinh(m\,x)\,m+\cosh(m\,x)\,m)$$

```
> bc1:=theta[b]-evalf(subs(x=0,rhs(Eq3)));
```

$$bc1 := \theta_b-1.\,\_C1-1.\,\_C2$$

```
> bc2:=simplify(subs(x=L,rhs(Eq4)));
```

$$bc2 := m\,(\_C1\,\sinh(m\,L)-\_C1\,\cosh(m\,L)+\_C2\,\sinh(m\,L)+\_C2\,\cosh(m\,L))$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \{\_C1 = \frac{0.5000000000\ \theta_b\,(\sinh(m\,L) + \cosh(m\,L))}{\cosh(m\,L)},$$

$$\_C2 = -\frac{0.5000000000\ \theta_b\,(\sinh(m\,L) - 1.\ \cosh(m\,L))}{\cosh(m\,L)}\}$$

```
> assign(const):
> sol1:=simplify(Eq3);
```

$$sol1 := \theta(x) = -\frac{1.\ \theta_b\,(\sinh(m\,L)\sinh(m\,x) - 1.\ \cosh(m\,L)\cosh(m\,x))}{\cosh(m\,L)}$$

```
> q:=simplify(-k*A[c]*subs(x=0,rhs(Eq4)));
```

$$q := \frac{k\,A_c\,\theta_b\,m\sinh(m\,L)}{\cosh(m\,L)}$$

As a numerical example, consider an aluminum alloy fin (k = 175W/m.K) 3mm wide, 0.4 mm thick, and 40mm long attached to a surface at 67°C. The fin operates in a convective environment at 27°C and the convective hear transfer coefficient is h = 8W/m^2-K. The temperature distribution in the fin, the heat dissipated by the fin and the fin tip temperature are sought.

```
> A[c]:=0.003*0.0004;P:=2*(0.003+0.0004);h:=8;k:=175;
m:=sqrt(h*P/(k*A[c]));L:=0.04;theta[b]:=67-27;
```

$$A_c := 0.12\ 10^{-5}$$

$$P := 0.0068$$

$$h := 8$$

$$k := 175$$

$$m := 16.09495632$$

$$L := 0.04$$

$$\theta_b := 40$$

```
> temp_distribution:=evalf(rhs(sol1),4);
```

$$temp\_distribution := -22.70\ \sinh(16.09\ x) + 40.00\ \cosh(16.09\ x)$$

```
> Heat_transfer_rate:=evalf(q,4);
```

$$Heat\_transfer\_rate := 0.07673$$

```
> plot(temp_distribution,x=0..0.04,color=black,labels=[x,
"Theta"]);
```

```
> Tip_temperature:=evalf(subs(x=0.04,rhs(sol1)),4;
```

$$Tip\_temperature := 32.92$$

The temperature distribution is shown in Fig.6.3.

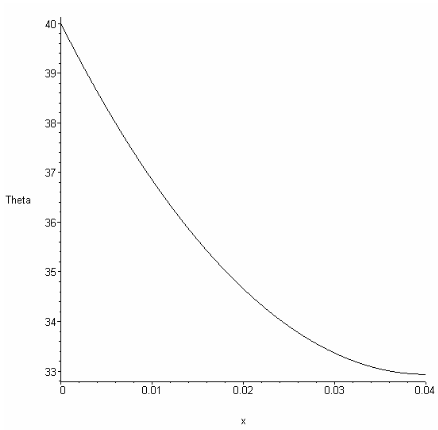


Fig. 6.3: Theta as a function of distance

Example 6.2: Fixed temperature at ends

The appropriate boundary conditions for this case are:

$$T(x=0)=T_1 \quad \theta(x=0)=\theta_1 \tag{6.7}$$

and

$$T(x=L)=T_2 \quad \theta(x=L)=\theta_2 \tag{6.8}$$

The heat transfer from the base surface where $T=T_1$ is

$$q_1 = -kA_c \left.\frac{dT}{dx}\right|_{x=0} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=0} \tag{6.9}$$

Similarly, the heat transfer from the base surface where $T=T_2$ is

$$q_2 = -kA_c \left.\frac{dT}{dx}\right|_{x=L} = -kA_c \left.\frac{d\theta}{dx}\right|_{x=L} \tag{6.10}$$

The total heat dissipated by the fin, q is the sum of q_1 and q_2.

$$q = q_2 + q_1 \tag{6.11}$$

```
> restart;
> Eq1:=diff(theta(x),x,x)-m^2*theta(x)=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\theta(x)\right) - m^2\,\theta(x) = 0$$

```
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\,e^{(-m\,x)} + \_C2\,e^{(m\,x)}$$

```
> Eq3:=convert(Eq2,trig);
```

$$Eq3 := \theta(x) = \_C1\,(\cosh(m\,x) - \sinh(m\,x)) + \_C2\,(\cosh(m\,x) + \sinh(m\,x))$$

```
> Eq4:=diff(Eq3,x);
```

$$Eq4 := \frac{d}{dx}\theta(x) = \_C1\,(\sinh(m\,x)\,m - \cosh(m\,x)\,m) + \_C2\,(\sinh(m\,x)\,m + \cosh(m\,x)\,m)$$

```
> bc1:=theta[1]-simplify(subs(x=0,rhs(Eq3)));
```

$$bc1 := \theta_1 - \_C1\,\cosh(0) + \_C1\,\sinh(0) - \_C2\,\cosh(0) - \_C2\,\sinh(0)$$

```
> bc2:=theta[2]-simplify(subs(x=L,rhs(Eq3)));
```

$$bc2 := \theta_2 - \_C1\,\cosh(m\,L) + \_C1\,\sinh(m\,L) - \_C2\,\cosh(m\,L) - \_C2\,\sinh(m\,L)$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \{\_C2 = -\frac{1}{2}\frac{-\theta_2 + \cosh(m\,L)\,\theta_1 - \sinh(m\,L)\,\theta_1}{\sinh(m\,L)},$$
$$\_C1 = \frac{1}{2}\frac{\sinh(m\,L)\,\theta_1 - \theta_2 + \cosh(m\,L)\,\theta_1}{\sinh(m\,L)}\}$$

```
> assign(const);
```

```
> sol2:=simplify(Eq3);
```

$$sol2 := \theta(x) = \frac{\sinh(m\,L)\,\theta_1\cosh(m\,x) + \theta_2\sinh(m\,x) - \cosh(m\,L)\,\theta_1\sinh(m\,x)}{\sinh(m\,L)}$$

```
> q1:=simplify(-k*A[c]*subs(x=0,rhs(Eq4)));
```

$$q1 := \frac{k\,A_c\,m\,(-\sinh(m\,L)\,\theta_1\sinh(0) - \theta_2\cosh(0) + \cosh(m\,L)\,\theta_1\cosh(0))}{\sinh(m\,L)}$$

```
> q2:=simplify(k*A[c]*subs(x=L,rhs(Eq4)));
```

$$q2 := -\frac{k\,A_c\,m\,(\theta_1 - \theta_2\cosh(m\,L))}{\sinh(m\,L)}$$

We use the foregoing analytical results to study the performance of a fin 50mm wide, 2mm thick and 60mm long attached to two plates, one maintained at 100°C while the temperature of the other plate changes from 20°C to 100°C in increments of 20°C. The top and bottom surfaces of the plate are cooled by air with a heat transfer coefficient of h = 20W/m^2-K. The thermal conductivity of the fin is k=50W/m-K.

```
> A[c]:=0.002*0.050;k:=50;theta[1]:=100-20;L:=0.06;
P:=2*(0.002+0.050); h:=20;m:=sqrt(h*P/k/A[c]);
```

$$A_c := 0.000100$$
$$k := 50$$
$$\theta_1 := 80$$
$$L := 0.06$$
$$P := 0.104$$
$$h := 20$$
$$m := 20.39607805$$

```
> y:=rhs(sol2);
```

$$y := 79.99999997\ \cosh(20.39607805\ x) + 0.6439476264\ \theta_2 \sinh(20.39607805\ x) - 95.15187177\ \sinh(20.39607805\ x)$$

```
> for n from 0 to 4 do
> y||n:=subs(theta[2]=20*n,y);od;
```

$$y0 := 79.99999997\ \cosh(20.39607805\ x) - 95.15187177\ \sinh(20.39607805\ x)$$
$$y1 := 79.99999997\ \cosh(20.39607805\ x) - 82.27291924\ \sinh(20.39607805\ x)$$
$$y2 := 79.99999997\ \cosh(20.39607805\ x) - 69.39396671\ \sinh(20.39607805\ x)$$
$$y3 := 79.99999997\ \cosh(20.39607805\ x) - 56.51501419\ \sinh(20.39607805\ x)$$
$$y4 := 79.99999997\ \cosh(20.39607805\ x) - 43.63606166\ \sinh(20.39607805\ x)$$

```
> plot([y||(0..4),5],x=0..0.06,labels=[x,'Theta'],
linestyle=[SOLID,DOT,DASH,DASHDOT,SOLID,DASH,DOT],
color=black,legend=["Theta[2]=0","Theta[2]=20",
"Theta[2]=40","Theta[2]=60","Theta[2]=80"," "]);
```

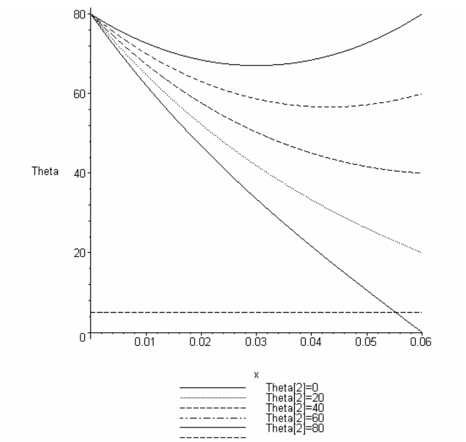


Fig. 6.4: Temperature distributions in the fin for various end temperatures.

```
> print(Theta[2],"q1","q2");
> for theta[2] from 0 by 20 to 80 do
print(theta[2],q1(theta[2]),q2(theta[2]),(q1(theta[2])+q2(th
eta[2])));od;
```

0, 9.703625011 , -5.253602417 , 4.450022594
20, 8.390224409 , -2.827696165 , 5.562528244
40, 7.076823801 , -0.4017899125 , 6.675033888
60, 5.763423199 , 2.024116338 , 7.787539537
80, 4.450022595 , 4.450022595 , 8.900045190

The first column gives θ_2, the second column gives q_1, the third column gives q_2 and the fourth column gives $q = q_1 + q_2$. These results clearly indicate that the heat dissipation from the fin is a maximum (q = 8.90 W) when the fin is heated symmetrically, that is, at $\theta_1 = \theta_2 = 80$°C. As θ_2 is reduced, q_1 increases but q_2 decreases. Note that q_2 eventually becomes negative, indicating that the right hand plate becomes a sink rather than a source of heat. The net result is a diminishing q. Heat dissipation from the fin becomes a minimum (q = 4.45W) when the temperature of the right hand plate is equal to the environmental temperature $\theta_2 = 0$. For the symmetrically heated fin, the minimum temperature occurs at the mid point of the fin. As θ_2 is reduced, this minimum gets closer to the right end plate and is eventually located at the right end plate when $\theta_2 = 0$.

Example 6.3: Specified heat fluxes at the ends

The appropriate boundary conditions for this case are:

$$x = 0, -k\frac{dT}{dx} = q_1 or \frac{d\theta}{dx} + \frac{q_1}{k} = 0 \tag{6.12}$$

$$x = L, -k\frac{dT}{dx} = q_2 or \frac{d\theta}{dx} + \frac{q_2}{k} = 0 \tag{6.13}$$

where q_1 and q_2 are the specified heat fluxes at x = 0 and x = L respectively. Once the temperature distribution in the fin is found, the temperatures at x = 0 and x = L can be readily found as

$$\theta(x=0) = \theta_1 = T_1 - T_a \tag{6.14}$$

and

$$\theta(x=L) = \theta_2 = T_2 - T_a \tag{6.15}$$

```
> restart;
```

```
> Eq1:=diff(theta(x),x,x)-m^2*theta(x)=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\theta(x)\right) - m^2\,\theta(x) = 0$$

```
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\, e^{(m\,x)} + \_C2\, e^{(-m\,x)}$$

```
> Eq3:=convert(Eq2,trig);
```

$$Eq3 := \theta(x) = \_C1\,(\cosh(m\,x) + \sinh(m\,x)) + \_C2\,(\cosh(m\,x) - \sinh(m\,x))$$

```
> Eq4:=diff(Eq3,x);
```

$$Eq4 := \frac{d}{dx}\theta(x) = \_C1\,(\sinh(m\,x)\,m + \cosh(m\,x)\,m) + \_C2\,(\sinh(m\,x)\,m - \cosh(m\,x)\,m)$$

```
> bc1:=evalf(simplify(subs(x=0,rhs(Eq4))))+q[1]/k;
```

$$bc1 := m\,(1.\,\_C1 - 1.\,\_C2) + \frac{q_1}{k}$$

```
> bc2:=simplify(subs(x=L,rhs(Eq4)))-q[2]/k;
```

$$bc2 := m\,(\_C1\,\sinh(m\,L) + \_C1\,\cosh(m\,L) + \_C2\,\sinh(m\,L) - \_C2\,\cosh(m\,L)) - \frac{q_2}{k}$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$const := \left\{\_C2 = \frac{e^{(m\,L)}\,(q_2 + e^{(m\,L)}\,q_1)}{m\,k\,(-1. + (e^{(m\,L)})^2)},\ \_C1 = \frac{q_2\,e^{(m\,L)} + q_1}{m\,k\,(-1. + (e^{(m\,L)})^2)}\right\}$$

```
> assign(const):
```

```
> sol3:=simplify(convert(Eq3,trig));
```

$$sol3 := \theta(x) = (q_2 \sinh(m\,L)\cosh(m\,x) + q_2 \cosh(m\,L)\cosh(m\,x) + q_1 \sinh(m\,x) + q_1 \cosh(m\,L)^2 \cosh(m\,x) - 1.\; q_1 \cosh(m\,L)^2 \sinh(m\,x) + \sinh(m\,L)\, q_1 \cosh(m\,L)\cosh(m\,x) - 1.\; \sinh(m\,L)\, q_1 \cosh(m\,L)\sinh(m\,x)) \Big/ (m\,k\,(-1. + \cosh(m\,L)^2 + \sinh(m\,L)\cosh(m\,L)))$$

```
> theta[1]:=simplify(subs(x=0,rhs(sol3)));
```

$$\theta_1 := \frac{q_2 \sinh(m\,L) + q_2 \cosh(m\,L) + q_1 \cosh(m\,L)^2 + \sinh(m\,L)\, q_1 \cosh(m\,L)}{m\,k\,(-1. + \cosh(m\,L)^2 + \sinh(m\,L)\cosh(m\,L))}$$

```
> theta[2]:=simplify(subs(x=L,rhs(sol3)));
```

$$\theta_2 := \frac{q_2 \sinh(m\,L)\cosh(m\,L) + q_2 \cosh(m\,L)^2 + q_1 \sinh(m\,L) + q_1 \cosh(m\,L)}{m\,k\,(-1. + \cosh(m\,L)^2 + \sinh(m\,L)\cosh(m\,L))}$$

6.3.2 Trapezoidal Fin

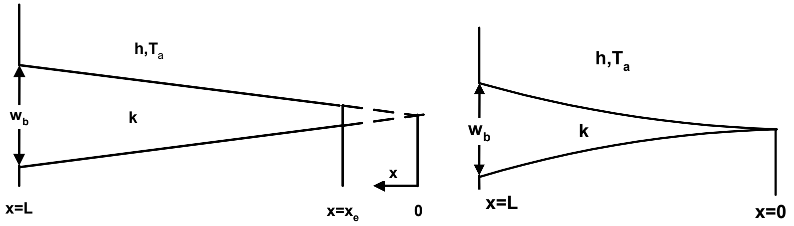


Fig 6.5 a,b: Trapezoidal and concave parabolic fin

For the trapezoidal fin shown in Fig 6.5a , the mathematics is greatly simplified if the origin of the x-coordinate is taken as the intersection of the sloping faces of the fin. This change of origin for x does not affect the general fin equation (6.1). However, A_c and A_s now become functions of x as follows

$$A_c = \frac{w_b x}{L}, A_s = 2(x - x_e)$$

where a unit depth has been assumed. In writing expressions for A_s, the fin is assumed to be sufficiently thin. Differentiation of A_s gives dA_s/dx = 2 and after substitution of A_c and dA_s/dx into Eq. (6.1). the differential equation for the temperature distribution in the trapezoidal fin is obtained as

$$x\frac{d^2T}{dx^2}+\frac{dT}{dx}-\frac{2hL}{kw_b}(T-T_a)=0 \qquad (6.16)$$

Introducing the variable $\theta = T - T_a$ and defining $m^2 = 2h/kw_b$, Eq. (6.16) becomes

$$\frac{d^2\theta}{dx^2}+\frac{d\theta}{dx}-m^2L\theta=0 \qquad (6.17)$$

We first create the differential equation (Eq. 6.17) and then solve it using the dsolve command to get a general solution. In order to obtain a solution in the terms of the modified Bessel functions of real arguments rather than imaginary arguments, it is necessary to let *Maple* know that L is a real and positive quantity. This is done by using the assume command. *Maple* then displays L as ~L. However, the tilde can be suppressed by invoking the command interface(show assumed=0). Once the general solution is developed, it is differentiated to obtain the derivative. Finally, solutions for the temperature distribution and heat transfer rates are developed for three sets of boundary conditions.

Example 6.4: Constant base temperature and convecting tip
The appropriate boundary conditions for this case are:

$$x = x_{e,}\, k\frac{dT}{dx}=h_t(T-T_a)=0 \qquad (6.18)$$

$$x = L, T = T_b or \theta - \theta_b = 0 \qquad (6.19)$$

where h_t is the tip heat transfer coefficient. The heat transfer from the fin can be obtained by applying the Fourier law at the base of the fin where
x = L

$$q = kw_b\left.\frac{d\theta}{dx}\right|_{x=L} \qquad (6.20)$$

```
> restart;
> Eq1:=x*diff(theta(x),x,x)+diff(theta(x),x)-m^2*L*
theta(x)=0;
```

$$Eq1 := x\left(\frac{d^2}{dx^2}\theta(x)\right)+\left(\frac{d}{dx}\theta(x)\right)-m^2L\,\theta(x)=0$$

```
> assume(L>0):Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\ \text{BesselI}(0, 2\, m \sqrt{L\sim}\, \sqrt{x}) + \_C2\ \text{BesselK}(0, 2\, m \sqrt{L\sim}\, \sqrt{x})$$

```
> Eq3:=diff(Eq2,x);
```

$$Eq3 := \frac{d}{dx}\theta(x) = \frac{\_C1\ \text{BesselI}(1, 2\, m \sqrt{L\sim}\, \sqrt{x})\, m \sqrt{L\sim}}{\sqrt{x}} - \frac{\_C2\ \text{BesselK}(1, 2\, m \sqrt{L\sim}\, \sqrt{x})\, m \sqrt{L\sim}}{\sqrt{x}}$$

```
> bc1:=subs(x=xe,rhs(Eq3))-h[t]/k*subs(x=xe,rhs(Eq2))=0;
```

$$bc1 := \frac{\_C1\ \text{BesselI}(1, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, m \sqrt{L\sim}}{\sqrt{xe}} - \frac{\_C2\ \text{BesselK}(1, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, m \sqrt{L\sim}}{\sqrt{xe}} - \frac{h_t\,(\_C1\ \text{BesselI}(0, 2\, m \sqrt{L\sim}\, \sqrt{xe}) + \_C2\ \text{BesselK}(0, 2\, m \sqrt{L\sim}\, \sqrt{xe}))}{k} = 0$$

```
> bc2:=subs(x=L,rhs(Eq2))-theta[b];
```

$$bc2 := \_C1\ \text{BesselI}(0, 2\, m\, L\sim) + \_C2\ \text{BesselK}(0, 2\, m\, L\sim) - \theta_b$$

```
> consts:=solve({bc1,bc2},{_C1,_C2}):
> assign(consts):
> simplify(Eq2);
```

$$\theta(x) = \theta_b\,(\text{BesselI}(0, 2\, m \sqrt{L\sim}\, \sqrt{x})\, \text{BesselK}(1, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, m \sqrt{L\sim}\, k$$
$$+ \text{BesselI}(0, 2\, m \sqrt{L\sim}\, \sqrt{x})\, h_t \sqrt{xe}\, \text{BesselK}(0, 2\, m \sqrt{L\sim}\, \sqrt{xe})$$
$$+ \text{BesselK}(0, 2\, m \sqrt{L\sim}\, \sqrt{x})\, \text{BesselI}(1, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, m \sqrt{L\sim}\, k$$
$$- \text{BesselK}(0, 2\, m \sqrt{L\sim}\, \sqrt{x})\, h_t \sqrt{xe}\, \text{BesselI}(0, 2\, m \sqrt{L\sim}\, \sqrt{xe}))\ \Big/\ ($$
$$\text{BesselI}(1, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, m \sqrt{L\sim}\, k\, \text{BesselK}(0, 2\, m\, L\sim)$$
$$+ \text{BesselK}(1, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, m \sqrt{L\sim}\, k\, \text{BesselI}(0, 2\, m\, L\sim)$$
$$- h_t \sqrt{xe}\, \text{BesselI}(0, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, \text{BesselK}(0, 2\, m\, L\sim)$$
$$+ h_t \sqrt{xe}\, \text{BesselK}(0, 2\, m \sqrt{L\sim}\, \sqrt{xe})\, \text{BesselI}(0, 2\, m\, L\sim))$$

```
> q:=simplify(k*w[b]*subs(x=L,rhs(Eq3)));
```

$$q := -k\, w_b\, \theta_b\, m\, (-\mathrm{BesselI}(1, 2\, m\, L\sim)\, \mathrm{BesselK}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, m\, \sqrt{L\sim}\, k - \mathrm{BesselI}(1, 2\, m\, L\sim)\, h_t\, \sqrt{xe}\, \mathrm{BesselK}(0, 2\, m\, \sqrt{L\sim}\, \sqrt{xe}) + \mathrm{BesselK}(1, 2\, m\, L\sim)\, \mathrm{BesselI}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, m\, \sqrt{L\sim}\, k - \mathrm{BesselK}(1, 2\, m\, L\sim)\, h_t\, \sqrt{xe}\, \mathrm{BesselI}(0, 2\, m\, \sqrt{L\sim}\, \sqrt{xe}))\,/\,(\mathrm{BesselI}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, m\, \sqrt{L\sim}\, k\, \mathrm{BesselK}(0, 2\, m\, L\sim) + \mathrm{BesselK}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, m\, \sqrt{L\sim}\, k\, \mathrm{BesselI}(0, 2\, m\, L\sim) - h_t\, \sqrt{xe}\, \mathrm{BesselI}(0, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, \mathrm{BesselK}(0, 2\, m\, L\sim) + h_t\, \sqrt{xe}\, \mathrm{BesselK}(0, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, \mathrm{BesselI}(0, 2\, m\, L\sim))$$

Example 6.5: Constant base temperature and insulated tip

This case can be recovered form Example 6.4 by simply putting $h_t = 0$. Eq4 and Eq5 below give the temperature distribution and heat transfer rate, respectively.

```
> Eq4:=limit(rhs(Eq2),h[t]=0);
```

$$Eq4 := (\theta_b\, \mathrm{BesselI}(0, 2\, m\, \sqrt{L\sim}\, \sqrt{x})\, \mathrm{BesselK}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe}) + \theta_b\, \mathrm{BesselK}(0, 2\, m\, \sqrt{L\sim}\, \sqrt{x})\, \mathrm{BesselI}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe}))\,/\,(\mathrm{BesselI}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, \mathrm{BesselK}(0, 2\, m\, L\sim) + \mathrm{BesselK}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, \mathrm{BesselI}(0, 2\, m\, L\sim))$$

```
> Eq5:=limit(q,h[t]=0);
```

$$Eq5 := (m\, \theta_b\, w_b\, k\, \mathrm{BesselI}(1, 2\, m\, L\sim)\, \mathrm{BesselK}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe}) - m\, \theta_b\, w_b\, k\, \mathrm{BesselK}(1, 2\, m\, L\sim)\, \mathrm{BesselI}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe}))\,/\,(\mathrm{BesselI}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, \mathrm{BesselK}(0, 2\, m\, L\sim) + \mathrm{BesselK}(1, 2\, m\, \sqrt{L\sim}\, \sqrt{xe})\, \mathrm{BesselI}(0, 2\, m\, L\sim))$$

Example 6.6: Convective base heating and insulated tip

Let the base of the fin be heated by a fluid at temperature, T_f. If the convective heat transfer coefficient is designated as h_f, the appropriate boundary conditions become

$$\left.\frac{dT}{dx}\right|_{x=xe} = 0 \quad or \quad \left.\frac{d\theta}{dx}\right|_{x=xe} = 0 \tag{6.21}$$

$$k\left.\frac{dT}{dx}\right|_{x=L} = h_f(T_f - T) \quad or \quad \left.\frac{d\theta}{dx}\right|_{x=L} = \frac{h_f}{k}(\theta_f - \theta) \tag{6.22}$$

The heat transfer from the fin can be found by computing the heat transfer by conduction at the base of the fin or by the convection heat transfer from the fluid to the base of the fin

$$q = kA_c\left.\frac{d\theta}{dx}\right|_{x=L} = h_f A_c(\theta_f - \theta_b) \tag{6.23}$$

```
> restart;
> Eq1:=x*diff(theta(x),x,x)+diff(theta(x),x)-m^2*L*
theta(x)=0;
```

$$Eq1 := x\left(\frac{d^2}{dx^2}\theta(x)\right) + \left(\frac{d}{dx}\theta(x)\right) - m^2 L\,\theta(x) = 0$$

```
> assume(L>0);
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\,\mathrm{BesselI}(0, 2\,m\sqrt{L\sim}\sqrt{x}) + \_C2\,\mathrm{BesselK}(0, 2\,m\sqrt{L\sim}\sqrt{x})$$

```
> Eq3:=diff(Eq2,x);
```

$$Eq3 := \frac{d}{dx}\theta(x) = \frac{\_C1\,\mathrm{BesselI}(1, 2\,m\sqrt{L\sim}\sqrt{x})\,m\sqrt{L\sim}}{\sqrt{x}} - \frac{\_C2\,\mathrm{BesselK}(1, 2\,m\sqrt{L\sim}\sqrt{x})\,m\sqrt{L\sim}}{\sqrt{x}}$$

```
> bc1:=subs(x=xe,rhs(Eq3));
```

$$bc1 := \frac{\_C1\,\mathrm{BesselI}(1, 2\,m\sqrt{L\sim}\sqrt{xe})\,m\sqrt{L\sim}}{\sqrt{xe}} - \frac{\_C2\,\mathrm{BesselK}(1, 2\,m\sqrt{L\sim}\sqrt{xe})\,m\sqrt{L\sim}}{\sqrt{xe}}$$

```
> bc2:=subs(x=L,rhs(Eq3))-h[f]/k*(theta[f]-subs(x=L,
rhs(Eq2)));
```

$$bc2 := \_C1\ \mathrm{BesselI}(1, 2\ m\ L\sim)\ m - \_C2\ \mathrm{BesselK}(1, 2\ m\ L\sim)\ m - \frac{h_f(\theta_f - \_C1\ \mathrm{BesselI}(0, 2\ m\ L\sim) - \_C2\ \mathrm{BesselK}(0, 2\ m\ L\sim))}{k}$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2}):
> assign(const):
> simplify(Eq2);
```

$$\theta(x) = h_f \theta_f (\mathrm{BesselK}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselI}(0, 2\ m \sqrt{L\sim} \sqrt{x}) + \mathrm{BesselI}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselK}(0, 2\ m \sqrt{L\sim} \sqrt{x})) \Big/ (-\mathrm{BesselI}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselK}(1, 2\ m\ L\sim)\ m\ k + \mathrm{BesselI}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ h_f \mathrm{BesselK}(0, 2\ m\ L\sim) + \mathrm{BesselK}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselI}(1, 2\ m\ L\sim)\ m\ k + \mathrm{BesselK}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ h_f \mathrm{BesselI}(0, 2\ m\ L\sim))$$

```
> q:=simplify(k*w[b]*subs(x=L,rhs(Eq3)));
```

$$q := k\ w_b\ h_f \theta_f m\ (-\mathrm{BesselK}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselI}(1, 2\ m\ L\sim) + \mathrm{BesselI}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselK}(1, 2\ m\ L\sim)) \Big/ (\mathrm{BesselI}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselK}(1, 2\ m\ L\sim)\ m\ k - \mathrm{BesselI}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ h_f \mathrm{BesselK}(0, 2\ m\ L\sim) - \mathrm{BesselK}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ \mathrm{BesselI}(1, 2\ m\ L\sim)\ m\ k - \mathrm{BesselK}(1, 2\ m \sqrt{L\sim} \sqrt{xe})\ h_f \mathrm{BesselI}(0, 2\ m\ L\sim))$$

6.3.3 Triangular Fin

Equation (6.17) is also applicable to a triangular fin when $x_e = 0$.

Example 6.7: Convective base heating
Consider the fin to be convectively heated at the base. Equation 6.22) provides one boundary condition and the condition of finite temperature at the tip provides the other.

```
> restart
> Eq1:=x*diff(theta(x),x,x)+diff(theta(x),x)-m^2*L*
theta(x)=0;
```

$$Eq1 := x\left(\frac{d^2}{dx^2}\theta(x)\right) + \left(\frac{d}{dx}\theta(x)\right) - m^2 L\,\theta(x) = 0$$

```
> assume(L>0):interface(showassumed=0):
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\ \mathrm{BesselI}(0, 2\,m\sqrt{L\sim}\sqrt{x}) + \_C2\ \mathrm{BesselK}(0, 2\,m\sqrt{L\sim}\sqrt{x})$$

```
> Eq3:=diff(Eq2,x);
```

$$Eq3 := \frac{d}{dx}\theta(x) = \frac{\_C1\ \mathrm{BesselI}(1, 2\,m\sqrt{L\sim}\sqrt{x})\,m\sqrt{L\sim}}{\sqrt{x}} - \frac{\_C2\ \mathrm{BesselK}(1, 2\,m\sqrt{L\sim}\sqrt{x})\,m\sqrt{L\sim}}{\sqrt{x}}$$

Because theta becomes infinite at x=0 (tip), the constant _C2 must be set to zero to ensure a finite temperature at the tip.

```
> _C2:=0;
```

$$\_C2 := 0$$

```
> bc:=subs(x=L,rhs(Eq3))-h[f]/k*(theta[f]-
subs(x=L,rhs(Eq2)));
```

$$bc := \_C1\ \mathrm{BesselI}(1, 2\,m\,L\sim)\,m - \frac{h_f(\theta_f - \_C1\ \mathrm{BesselI}(0, 2\,m\,L\sim))}{k}$$

```
> _C1:=solve(bc,_C1);
```

$$\_C1 := \frac{h_f\,\theta_f}{\mathrm{BesselI}(1, 2\,m\,L\sim)\,m\,k + h_f\,\mathrm{BesselI}(0, 2\,m\,L\sim)}$$

```
> Eq4:=Eq2;
```

$$Eq4 := \theta(x) = \frac{h_f\,\theta_f\,\mathrm{BesselI}(0, 2\,m\sqrt{L\sim}\sqrt{x})}{\mathrm{BesselI}(1, 2\,m\,L\sim)\,m\,k + h_f\,\mathrm{BesselI}(0, 2\,m\,L\sim)}$$

```
> Eq5:=diff(Eq4,x);
```

$$Eq5 := \frac{d}{dx}\theta(x) = \frac{h_f\,\theta_f\,\mathrm{BesselI}(1, 2\,m\sqrt{L\sim}\sqrt{x})\,m\sqrt{L\sim}}{(\mathrm{BesselI}(1, 2\,m\,L\sim)\,m\,k + h_f\,\mathrm{BesselI}(0, 2\,m\,L\sim))\sqrt{x}}$$

```
> q:=k*A[c]*subs(x=L,rhs(Eq5));
```

$$q := \frac{k\,A_c\,h_f\,\theta_f\,\mathrm{BesselI}(1, 2\,m\,L\sim)\,m}{\mathrm{BesselI}(1, 2\,m\,L\sim)\,m\,k + h_f\,\mathrm{BesselI}(0, 2\,m\,L\sim)}$$

To illustrate the results derived previously, let us find the temperature distribution, heat transfer rate, and temperature at the base and the tip of the fin for a triangular fin with k = 50 W/m-K. The fin is 2mm thick at the base, 60mm long and 30 mm wide. The base of the fin is heated by a fluid at a temperature of 100°C with a heat transfer coefficient of 100 W/m^2-K. The fin rejects heat to the surroundings at a temperature of 20°C with a heat transfer coefficient of 10W/m^2-K.

```
> k:=50;h[a]:=10;h[f]:=100;T[a]:=20;
T[f]:=100;theta[f]:=T[f]-T[a];
Ll:=0.06;A[c]:=0.002*0.03;P:=2*(0.002+0.03);
m:=sqrt(h[a]*P/k/A[c]);
```

$$k := 50$$
$$h_a := 10$$
$$h_f := 100$$
$$T_a := 20$$
$$T_f := 100$$
$$\theta_f := 80$$
$$Ll := 0.06$$
$$A_c := 0.00006$$
$$P := 0.064$$
$$m := 14.60593487$$

```
> Temp_Distribution:=T[a]+simplify(subs(L=Ll,rhs(Eq4)));
```

$$Temp\_Distribution := 20 + 7.193105765\ \mathrm{BesselI}(0., 7.155417530\ \sqrt{x})$$

```
> Heat_Transfer_Rate:=simplify(subs(L=Ll,q));
```

$$Heat\_Transfer\_Rate := 0.3967615881$$

```
> Base_temperature:=subs(x=0.06,Temp_Distribution):evalf(%);
```

$$33.87306866$$

```
> Tip_Temperature:=subs(x=0,Temp_Distribution):evalf(%);
```

$$27.19310576$$

6.3.4 Concave Parabolic Fin

For the concave parabolic fin shown in Fig 6.5 b , with a unit width

$$A_c = w_b\left(\frac{x}{b}\right)^{1/2}$$

and if the fin is sufficiently thin, $A_s = 2x$ and $dA_s/dx = 2$. Equation (6.1) in this case reduces to

$$x^2\frac{d^2T}{dx^2} + 2x\frac{dT}{dx} - \frac{2hL^2}{kw_b}(T - T_a) = 0 \tag{6.24}$$

Introducing the temperature excesses, $\theta = T - T_a$, and defining $m^2 = 2h/kw_b$, Eq. (6.24) can be expressed as

$$x^2\frac{d^2\theta}{dx^2} + 2x\frac{d\theta}{dx} - m^2L^2\theta = 0 \tag{6.25}$$

Example 6.8: Convective base heating
Equation (6.22) which describes the convective heating at the base gives one boundary condition and the condition of finite tip temperature provides the second. Equation (6.23) gives the heat transfer rate.

```
> restart;
> Eq1:=x^2*diff(theta(x),x,x)+2*x*diff(theta(x),x)-(m*l)^2*
theta(x)=0;
```

$$Eq1 := x^2\left(\frac{d^2}{dx^2}\theta(x)\right) + 2\,x\left(\frac{d}{dx}\theta(x)\right) - m^2\,l^2\,\theta(x) = 0$$

```
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\,x^{\left(-1/2 + \frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)} + \_C2\,x^{\left(-1/2 - \frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}$$

```
> Eq3:=diff(Eq2,x);
```

$$Eq3 := \frac{d}{dx}\theta(x) = \frac{\_C1\, x^{\left(-1/2+\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}\left(-\frac{1}{2}+\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}{x} + \frac{\_C2\, x^{\left(-1/2-\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}\left(-\frac{1}{2}-\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}{x}$$

Because the second term in Eq2 becomes unbounded at x=0, we set _C2 = 0 to ensure a bounded solution.

```
> _C2:=0;
```

$$\_C2 := 0$$

```
> bc:=subs(x=L,rhs(Eq3))-h[f]/k*(theta[f]-
subs(x=L,rhs(Eq2)));
```

$$bc := \frac{\_C1\, L^{\left(-1/2+\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}\left(-\frac{1}{2}+\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}{L} - \frac{h_f\left(\theta_f - \_C1\, L^{\left(-1/2+\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}\right)}{k}$$

```
> _C1:=solve(bc,_C1):
> Eq4:=simplify((Eq2));
```

$$Eq4 := \theta(x) = \frac{2\,h_f\,L^{\left(3/2-\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}\,\theta_f\,x^{\left(-1/2+\frac{\sqrt{1+4\,m^2\,l^2}}{2}\right)}}{-k+k\sqrt{1+4\,m^2\,l^2}+2\,h_f\,L}$$

```
> q:=simplify(h[f]*A[c]*(theta[f]-subs(x=L,rhs(Eq4))));
```

$$q := \frac{k\left(-1+\sqrt{1+4\,m^2\,l^2}\right)h_f\,A_c\,\theta_f}{-k+k\sqrt{1+4\,m^2\,l^2}+2\,h_f\,L}$$

SPINES

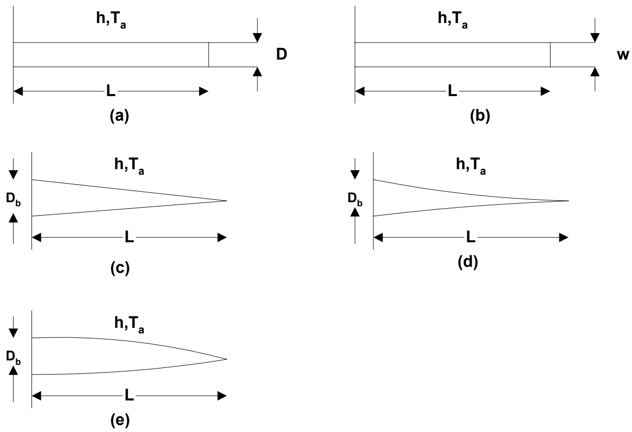


Fig. 6.6: Spines

The general fin equation (Eq. (6.1)) is also applicable to spines. The extended surface literature commonly treats five different spine profiles: cylindrical, rectangular, conical, concave parabolic, and convex parabolic. These profiles are shown in Fig. 6.6. Because the analysis of the governing equation is the same as that for straight fins, we abbreviate our discussion by presenting the analysis for only the conical spine. Then, we show how *Maple* can be effectively employed to generate tables and graphs of the efficiency for spines of all five profiles.

6.4.1 Conical Spine

For the conical spine shown in Fig 6.6 c and with x measured from the tip of the spine,

$$A_c = \frac{\pi}{4} D_b^2 \left(\frac{x}{L}\right)^2$$

and for a thin spine

$$A_s = (\pi D/2)x$$

With these substitutions, Eq. (6.1) reduces to

$$x^2 \frac{\partial^2 \theta}{\partial x^2} + 2x\frac{d\theta}{dx} - M^2 x\theta = 0 \qquad (6.26)$$

where

$$\theta = T - T_a \quad and \quad M^2 = \frac{4hL}{kD_b}$$

Example 6.9: Constant Base Temperature
One boundary condition is given by Eq. (6.19) and the condition of finite tip temperature provides the second.

```
> restart:
> Eq1:=x^2*diff(theta(x),x,x)+2*x*diff(theta(x),x)-M^2*x*
theta(x)=0;
```

$$Eq1 := x^2 \left(\frac{d^2}{dx^2}\theta(x)\right) + 2\,x\left(\frac{d}{dx}\theta(x)\right) - M^2\,x\,\theta(x) = 0$$

```
> assume(L>0):interface(showassumed=0);
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \frac{\_C1\ \mathrm{BesselI}(1, 2\,M\sqrt{x})}{\sqrt{x}} + \frac{\_C2\ \mathrm{BesselK}(1, 2\,M\sqrt{x})}{\sqrt{x}}$$

We set _C2 = 0 to ensure that there is a finite solution at x = 0 (tip).

```
> _C2:=0;
```

$$\_C2 := 0$$

```
> Eq3:=simplify(diff(Eq2,x));
```

$$Eq3 := \frac{d}{dx}\theta(x) = \frac{\_C1\ (-\mathrm{BesselI}(1, 2\,M\sqrt{x}) + \sqrt{x}\ \mathrm{BesselI}(0, 2\,M\sqrt{x})\,M)}{x^{(3/2)}}$$

```
> bc:=theta[b]-subs(x=L,rhs(Eq2));
```

$$bc := \theta_b - \frac{\_C1\ \mathrm{BesselI}(1, 2\, M \sqrt{L\sim})}{\sqrt{L\sim}}$$

```
> _C1:=solve(bc=0,_C1);
```

$$\_C1 := \frac{\theta_b \sqrt{L\sim}}{\mathrm{BesselI}(1, 2\, M \sqrt{L\sim})}$$

```
> Eq4:=Eq2;
```

$$Eq4 := \theta(x) = \frac{\theta_b \sqrt{L\sim}\ \mathrm{BesselI}(1, 2\, M \sqrt{x})}{\mathrm{BesselI}(1, 2\, M \sqrt{L\sim}) \sqrt{x}}$$

```
> q:=k*A[c]*subs(x=L,rhs(Eq3));
```

$$q := \frac{k\, A_c\, \theta_b\, (-\mathrm{BesselI}(1, 2\, M \sqrt{L\sim}) + \sqrt{L\sim}\ \mathrm{BesselI}(0, 2\, M \sqrt{L\sim})\, M)}{L\sim\ \mathrm{BesselI}(1, 2\, M \sqrt{L\sim})}$$

6.4.2 Efficiencies of Spines

Fig 6.6 shows the dimensions of the common spines to be considered. Each spine has its base temperature maintained at T_b. For the cylindrical and rectangular spines, which are spines of constant cross sectional areas, the tip is assumed to be insulated and for the conical, concave parabolic, and convex parabolic spines we require that the tip temperature be finite. Spines operating under these conditions have been analyzed in Kraus, Aziz and Welty [1] and they give expressions for the fin efficiencies.

6.4.2.1 Cylindrical and Rectangular Spines

$$\eta_1 = \frac{\tanh(mL)}{mL} \tag{6.27}$$

where m^2 = 4h/kD for cylindrical spines and m = 2h(w+b)/kw for the rectangular spine.

6.4.2.2 Conical Spine

$$\eta_2 = \frac{\sqrt{2}I_2(2\sqrt{2mL})}{mLI_1(2\sqrt{2mL})} \tag{6.28}$$

where $m^2 = 2h/kD_b$

6.4.2.3 Concave Parabolic Spine

$$\eta_3 = \frac{2}{\left(1+\frac{8}{9}(mL)^2\right)^{1/2}} \tag{6.29}$$

where $m^2 = 2h/kD_b$

6.4.2.4 Convex Parabolic Spine

$$\eta_4 = \frac{3}{2\sqrt{2}}\frac{I_1(\frac{4}{3}\sqrt{2mL})}{mLI_0(\frac{4}{3}\sqrt{2mL})} \tag{6.30}$$

Equations (6.27) through (6.30) show that the fin efficiency is a function of the parameter, mL. In example 6.10, we create η_1, η_2, η_3 and η_4 as functions of mL. Next we write a do loop to generate a table of efficiencies for mL ranging from 0.20 to 2.00. Finally, a single plot is created showing the efficiencies of the various spines as a function of mL (Fig.6.7). The efficiency plot shows that the spine of concave parabolic profile has the highest efficiency. Although the conical spine exhibits the second highest efficiency, its efficiency is significantly lower than that of a concave parabolic spine. The efficiency of the cylindrical and rectangular spines trail behind the efficiency of the conical spine, while the convex parabolic spine has the lowest efficiency of all five profiles.

Example 6.10

- **Rectangular or Cylindrical**:

```
> restart;
> eta[1]:=mL->tanh(mL)/mL;
```

$$\eta_1 := mL \rightarrow \frac{\tanh(mL)}{mL}$$

```
> for k from 0.2 by 0.2 to 2 do print(k,eta[1](k));od;
```

0.2, 0.9868766010
0.4, 0.9498724058
0.6, 0.8950826117
0.8, 0.8300459629
1.0, 0.7615941560
1.2, 0.6947121725
1.4, 0.6323940344
1.6, 0.5760428465
1.8, 0.5260033404
2.0, 0.4820137900

- **Conical:**

```
> eta[2]:=mL->
sqrt(2)/mL*BesselI(2,2*sqrt(2)*mL)/BesselI(1,2*sqrt(2)*mL);;
```

$$\eta_2 := mL \rightarrow \frac{\sqrt{2}\ \mathrm{BesselI}(2, 2\sqrt{2}\ mL)}{mL\ \mathrm{BesselI}(1, 2\sqrt{2}\ mL)}$$

```
> for k from 0.2 by 0.2 to 2 do
print(k,evalf(eta[2](k)));od;
```

0.2, 0.9869277645
0.4, 0.9505982725
0.6, 0.8981290973
0.8, 0.8376241691
1.0, 0.7756355884
1.2, 0.7163113970
1.4, 0.6617224877
1.6, 0.6125620754
1.8, 0.5687600012
2.0, 0.5298905135

- **Concave Parabolic:**

```
> eta[3]:=mL->2/(1+sqrt(1+8/9*mL^2));
```

$$\eta_3 := mL \rightarrow \frac{2}{1+\sqrt{1+\frac{8}{9}mL^2}}$$

```
> for k from 0.2 by 0.2 to 2 do print(k,eta[3](k));od;
```

0.2, 0.9912657090
0.4, 0.9667683208
0.6, 0.9307033084
0.8, 0.8878814596
1.0, 0.8423292192
1.2, 0.7968232610
1.4, 0.7530217882
1.6, 0.7117818704
1.8, 0.6734524726
2.0, 0.6380857944

- **Convex Parabolic**

```
> eta[4]:=mL-> 3/(2*sqrt(2))*BesselI(1,4*sqrt(2)/3*mL)/mL/
BesselI(0,4*sqrt(2)/3*mL);
```

$$\eta_4 := mL \rightarrow \frac{3}{2}\frac{\text{BesselI}\left(1, \frac{4}{3}\sqrt{2}\ mL\right)}{\sqrt{2}\ mL\ \text{BesselI}\left(0, \frac{4}{3}\sqrt{2}\ mL\right)}$$

```
> for k from 0.2 by 0.2 to 2 do
print(k,evalf(eta[4](k)));od;
```

0.2, 0.9826335671
0.4, 0.9350309996
0.6, 0.8679834316
0.8, 0.7931432302
1.0, 0.7191984183

1.2, 0.6510079850
1.4, 0.5904782784
1.6, 0.5377768847
1.8, 0.4922499159
2.0, 0.4529608766

```
> plot([eta[1](mL),eta[2](mL),eta[3](mL),eta[4](mL)],
mL=0..4, labels=["mL","eta"], color=BLACK,legend=["eta 1",
"eta 2", "eta 3","eta 4"] ,linestyle=[DOT, SOLID, DASH,
DASHDOT]);
```

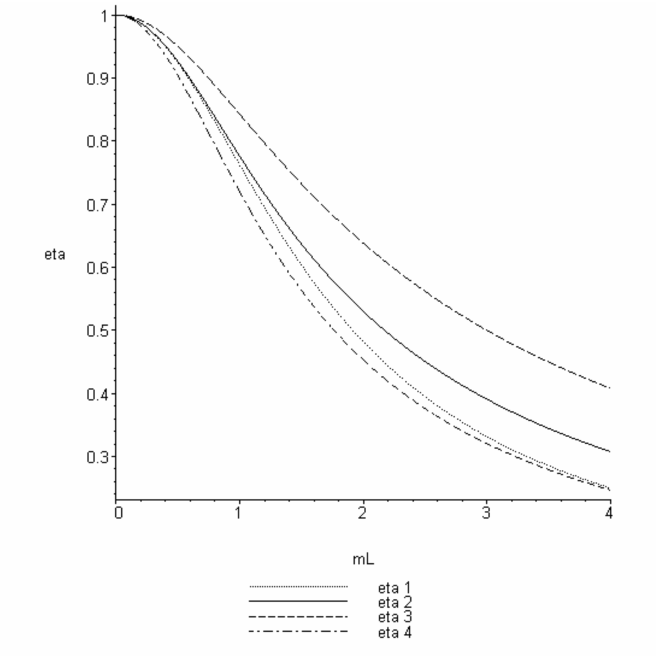


Fig. 6.7: Efficiencies for various fin geometries

6.5 ANNULAR FIN OF RECTANGULAR PROFILE

Heat transfer from the outside surface of a circular tube can be enhanced by attaching an annular (circular) fin as illustrated in Fig 6.8. The fin has a radius, r_b, at the base, a radius, r_t, at the tip and a uniform thickness, w. Replacing x by r in Eq. (6.1) and noting that $A_c = 2\pi rw$ and $A_s = 2\pi(r^2 - r_b^2)$, the general fin equation reduces to

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kw}(T - T_a) = 0 \tag{6.31}$$

with $m^2 = 2h/kw$ and $\theta = (T-T_a)$, Eq. (6.31) becomes

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0 \tag{6.32}$$

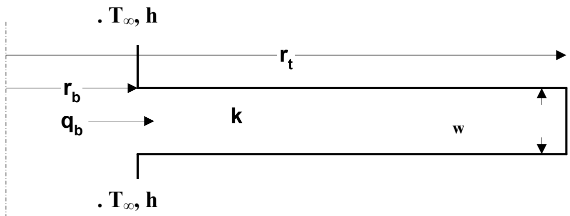


Fig. 6.8: Annular fin on circular tube.

Equation (6.32) is a modified Bessel equation of order zero which can be solved if two boundary conditions are specified. The solution for θ can then be utilized to obtain the heat transfer rate, q, as

$$q = -k(2\pi r_b w)\left.\frac{d\theta}{dr}\right|_{r=r_b} \tag{6.33}$$

Example 6.11: Convective heating at the base and convective cooling at the tip

The base of the fin is assumed to be heated by a fluid at T_f via convective heat transfer with a coefficient of h_f. The convective boundary conditions at the base of the fin can be expressed as

$$-k\left.\frac{dT}{dr}\right|_{r=r_b} = h_f(T_f - T) \quad or \quad h_f(\theta_f - \theta) + k\left.\frac{d\theta}{dr}\right|_{r=r_b} = 0 \tag{6.34}$$

Allowance for heat transfer from the fin tip requires that the boundary condition be written as

$$-k\left.\frac{dT}{dr}\right|_{r=r_t} = h_f(T-T_a) \quad or \quad h_t\theta + k\left.\frac{d\theta}{dr}\right|_{r=r_t} = 0 \tag{6.35}$$

where h_t is the convection heat transfer coefficient at the fin tip.

The ideal heat transfer, q_i, is the convection heat loss when the entire fin including the tip is kept at temperature T_b. Thus

$$q_{ideal} = h2\pi(r_t^2 - r_b^2)(T_b - T_a) + h_t 2\pi r_t w(T_b - T_a) \tag{6.36}$$

The ratio of q/q_{ideal} is the fin efficiency η.

```
> restart;
> Eq1:=diff(theta(r),r,r)+1/r*diff(theta(r),r)-m^2*
theta(r)=0;
```

$$Eq1 := \left(\frac{d^2}{dr^2}\theta(r)\right) + \frac{\frac{d}{dr}\theta(r)}{r} - m^2\,\theta(r) = 0$$

```
> Eq2:=dsolve(Eq1,theta(r));
```

$$Eq2 := \theta(r) = \_C1\ \mathrm{BesselI}(0, m\,r) + \_C2\ \mathrm{BesselK}(0, m\,r)$$

```
> Eq3:=diff(Eq2,r);
```

$$Eq3 := \frac{d}{dr}\theta(r) = \_C1\ \mathrm{BesselI}(1, m\,r)\,m - \_C2\ \mathrm{BesselK}(1, m\,r)\,m$$

```
> Eq4:=subs(r=r[b],rhs(Eq3));
```

$$Eq4 := \_C1\ \mathrm{BesselI}(1, m\,r_b)\,m - \_C2\ \mathrm{BesselK}(1, m\,r_b)\,m$$

```
> bc1:=h[f]*(theta[f]-subs(r=r[b],rhs(Eq2)))+k*subs(r=r[b],
rhs(Eq3));
```

$$bc1 := h_f(\theta_f - \_C1\ \mathrm{BesselI}(0, m\,r_b) - \_C2\ \mathrm{BesselK}(0, m\,r_b))$$
$$+ k\,(\_C1\ \mathrm{BesselI}(1, m\,r_b)\,m - \_C2\ \mathrm{BesselK}(1, m\,r_b)\,m)$$

```
> bc2:=h[t]*subs(r=r[t],rhs(Eq2))+k*subs(r=r[t],rhs(Eq3));
```

$$bc2 := h_t(\_C1\ \mathrm{BesselI}(0, m\,r_t) + \_C2\ \mathrm{BesselK}(0, m\,r_t))$$
$$+ k\,(\_C1\ \mathrm{BesselI}(1, m\,r_t)\,m - \_C2\ \mathrm{BesselK}(1, m\,r_t)\,m)$$

```
> const:=solve({bc1=0,bc2=0},{_C1,_C2}):
> assign(const):
> Eq5:=simplify(Eq2);
```

$$Eq5 := \theta(r) = h_f \theta_f (\mathrm{BesselI}(0, m\, r)\, h_t\, \mathrm{BesselK}(0, m\, r_t) - \mathrm{BesselI}(0, m\, r)\, k\, \mathrm{BesselK}(1, m\, r_t)\, m - \mathrm{BesselK}(0, m\, r)\, h_t\, \mathrm{BesselI}(0, m\, r_t) - \mathrm{BesselK}(0, m\, r)\, k\, \mathrm{BesselI}(1, m\, r_t)\, m) \Big/ (-h_t\, \mathrm{BesselI}(0, m\, r_t)\, h_f\, \mathrm{BesselK}(0, m\, r_b) - h_t\, \mathrm{BesselI}(0, m\, r_t)\, k\, \mathrm{BesselK}(1, m\, r_b)\, m - k\, \mathrm{BesselI}(1, m\, r_t)\, m\, h_f\, \mathrm{BesselK}(0, m\, r_b) - k^2\, \mathrm{BesselI}(1, m\, r_t)\, m^2\, \mathrm{BesselK}(1, m\, r_b) + h_f\, \mathrm{BesselI}(0, m\, r_b)\, h_t\, \mathrm{BesselK}(0, m\, r_t) - h_f\, \mathrm{BesselI}(0, m\, r_b)\, k\, \mathrm{BesselK}(1, m\, r_t)\, m - k\, \mathrm{BesselI}(1, m\, r_b)\, m\, h_t\, \mathrm{BesselK}(0, m\, r_t) + k^2\, \mathrm{BesselI}(1, m\, r_b)\, m^2\, \mathrm{BesselK}(1, m\, r_t))$$

```
> q:=simplify(-k*2*Pi*r[b]*w*Eq4);
```

$$q := -2\, k\, \pi\, r_b\, w\, h_f\, \theta_f\, m\, (\mathrm{BesselI}(1, m\, r_b)\, h_t\, \mathrm{BesselK}(0, m\, r_t) - \mathrm{BesselI}(1, m\, r_b)\, k\, \mathrm{BesselK}(1, m\, r_t)\, m + \mathrm{BesselK}(1, m\, r_b)\, h_t\, \mathrm{BesselI}(0, m\, r_t) + \mathrm{BesselK}(1, m\, r_b)\, k\, \mathrm{BesselI}(1, m\, r_t)\, m) \Big/ (-h_t\, \mathrm{BesselI}(0, m\, r_t)\, h_f\, \mathrm{BesselK}(0, m\, r_b) - h_t\, \mathrm{BesselI}(0, m\, r_t)\, k\, \mathrm{BesselK}(1, m\, r_b)\, m - k\, \mathrm{BesselI}(1, m\, r_t)\, m\, h_f\, \mathrm{BesselK}(0, m\, r_b) - k^2\, \mathrm{BesselI}(1, m\, r_t)\, m^2\, \mathrm{BesselK}(1, m\, r_b) + h_f\, \mathrm{BesselI}(0, m\, r_b)\, h_t\, \mathrm{BesselK}(0, m\, r_t) - h_f\, \mathrm{BesselI}(0, m\, r_b)\, k\, \mathrm{BesselK}(1, m\, r_t)\, m - k\, \mathrm{BesselI}(1, m\, r_b)\, m\, h_t\, \mathrm{BesselK}(0, m\, r_t) + k^2\, \mathrm{BesselI}(1, m\, r_b)\, m^2\, \mathrm{BesselK}(1, m\, r_t))$$

Special cases of the general solution developed here can be derived quite easily. For example, the case of constant base temperature, T_b, and insulated tip, which is often quoted in the heat transfer textbooks, can be obtained by putting $h_t = 0$, $\theta_f = \theta_b$ and allowing h_f to approach infinity. We may also change the boundary conditions and obtain the corresponding solutions.

The general solution is now used to obtain the temperature distribution, the heat transfer rate, the base temperature, the tip temperature, and the fin efficiency of an annular fin. The fin is attached to a circular tube having an outer diameter of 50mm. The tube carries a hot fluid at a temperature of 200°C and the heat transfer coefficient between the fluid and the inside tube surface is h_f = 100W/m^2-K. The conduction resistance of the tube may be neglected. The annular fin has a thermal conductivity of k = 40W/m-K and is 400 thick and 15mm long. The system operates in ambient air at a temperature of 20°C and the surface convection coefficient is h = 40W/m^2-K. The convection heat transfer coefficient for the tip is h_t = 50W/m^2-K.

```
> r[b]:=0.025;r[t]:=0.025+0.015;h[a]:=50;h[f]:=500;h[t]:=50;
w:=0.004;k:=40;m:=sqrt(2*h[a]/(k*w));T[a]:=20;T[f]:=100;thet
a[f]:=T[f]-T[a];
```

$$r_b := 0.025$$
$$r_t := 0.040$$
$$h_a := 50$$
$$h_f := 500$$
$$h_t := 50$$
$$w := 0.004$$
$$k := 40$$
$$m := 25.00000000$$
$$T_a := 20$$
$$T_f := 100$$
$$\theta_f := 80$$

```
> Temp_Distribution:=rhs(Eq5)+T[a];
```

$$Temp\_Distribution := 20.42318929\ \mathrm{BesselI}(0, 25.00000000\ r) + 22.09705388\ \mathrm{BesselK}(0, 25.00000000\ r) + 20$$

```
> Heat_Transfer_Rate:=evalf(q,4);
```

$$Heat\_Transfer\_Rate := 12.89$$

```
> Base_Temp:=evalf(subs(r=0.025,Temp_Distribution),4);
```

$$Base\_Temp := 58.94$$

```
> Tip_Temp:=evalf(subs(r=r[t],Temp_Distribution),4);
```

$$Tip\_Temp := 55.15$$

```
> q_ideal:=evalf(2*h[a]*Pi*(r[t]^2-r[b]^2)*(Base_Temp-
T[a])+h[t]*2*Pi*r[t]*w*(Base_Temp-T[a]));
```

$$q\_ideal := 13.88486564$$

```
> eta:=evalf(Heat_Transfer_Rate/q_ideal,4);
```

$$\eta := 0.9287$$

```
> plot(Temp_Distribution,r=0.025..0.040,labels=["r","T"],
color=BLACK);
```

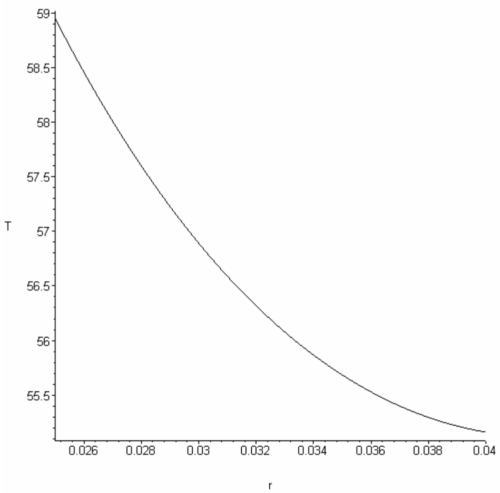


Fig 6.9: Temperature T °C versus radius in m for an annular fin

Fig. 6.9 shows the temperature distribution in the annular fin.

6.6 FINNED ARRAYS

Two fins in cascade are often used as building blocks to construct a complex finned array. Kraus, Aziz and Welty [1] provide a comprehensive treatment of finned arrays which are used extensively in the design of electronic equipment.

6.6.1 Cascaded Rectangular-Triangular Fin

An example of two fins in cascade is shown in Fig 6.10 has been analyzed and discussed in detail by Aziz[3]. It consists of a triangular fin of length L_1, attached to a rectangular fin. The overall length of the cascaded fin entity is L. The thickness of the rectangular fin is w and the assembly is attached to a primary surface at temperature T_b and operates in a convective environment at T_a. The surface heat transfer coefficient is h and both fins are constructed of a material having a thermal conductivity, k. The depth of the fin assembly is taken to be one meter.

For the rectangular part of the assembly, assuming the fin thickness, w, ito be very much smaller than the depth, A_c = w and P = 2. With these substitutions, Eq. (6.3) applies. For the triangular part, Eq. (6.16) is applicable with w replaced by w_b.

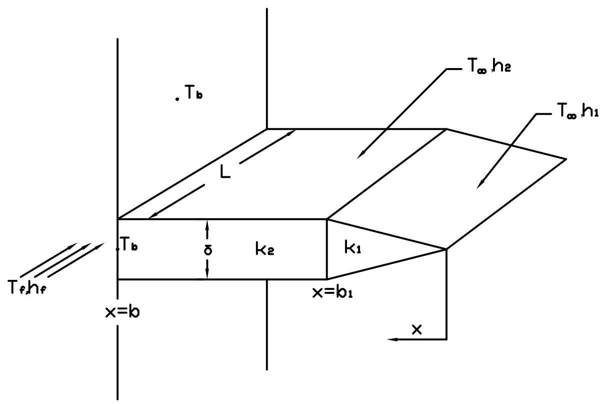


Fig 6.10: Cascaded Rectangular-Triangular Fin

To facilitate the solutions of Eq. (6.3) and (6.16), the following dimensionless parameters may be introduced

$$\theta = \frac{(T - T_a)}{(T_b - T_a)}, \qquad X = \frac{x}{L}$$

$$N_2^2 = \frac{2hL^2}{kw}, \qquad f = \frac{L_1}{L}$$

$$N_1^2 = fN_2^2$$

Denote the temperatures in the triangular and rectangular parts by θ_1 and θ_2 , respectively. Then with these temperatures and the dimensionless quantities substituted into eqs (6.16) and (6.3), the result is

$$X\frac{d^2\theta_1}{dX^2}+\frac{d\theta}{dX}-N_1^2\theta_1=0,0\le X\le f \tag{6.37}$$

and

$$\frac{d^2\theta_2}{dX^2}-N_2^2\theta=0, f\le X\le 1 \tag{6.38}$$

Two boundary conditions for eqs (6.37) and (6.38) can be generated by imposing the conditions of finite temperature at the tip of the triangular fin and T = T_b or θ_2 = 1 at the base of the rectangular fin.

$$\theta_1(X=0) \qquad \textit{finite} \tag{6.39}$$

and

$$\theta_2(X=1)=1 \tag{6.40}$$

Assuming perfect contact at the interface between the rectangular and triangular fins, we have two other boundary conditions. The first one to satisfy the condition of temperature continuity at X=f which gives

$$\theta_1(X=f)=\theta_1=\theta_2 \qquad or \qquad \theta_1-\theta_2=0 \tag{6.41}$$

and the second one to satisfy the heat flux continuity at X=f which gives

$$X=f,\quad \frac{d\theta_1}{dX}=\frac{d\theta_2}{dX} \quad or \quad \frac{d\theta_1}{dX}-\frac{d\theta_2}{dX}=0 \tag{6.42}$$

The solution for θ_2 can be used to compute the fin heat transfer

$$q=kw\left.\frac{dT}{dr}\right|_{x=L} \quad or \quad q=\frac{kw(T_b-T_c)}{L}\left.\frac{d\theta_1}{dX}\right|_{X=1} \tag{6.43}$$

In example 6.12, eqs (6.37) to (6.42) are solved to obtain temperature distributions θ_1 and θ_2. Expressions for the heat transfer, q, and the interface temperature, θ_i , which equals $\theta_1(f)$ or $\theta_2(f)$ are also derived.

Example 6.12

```
> restart;
> Eq1:=X*diff(theta[1](X),X,X)+diff(theta[1](X),X)-N[1]^2*theta[1](X)=0;
```

$$Eq1 := X\left(\frac{d^2}{dX^2}\theta_1(X)\right)+\left(\frac{d}{dX}\theta_1(X)\right)-N_1^{\,2}\,\theta_1(X)=0$$

```
> Eq2:=dsolve(Eq1,theta[1](X));
```

$$Eq2 := \theta_1(X) = \_C1\ \mathrm{BesselI}(0, 2\,N_1\sqrt{X}) + \_C2\ \mathrm{BesselK}(0, 2\,N_1\sqrt{X})$$

For finite tip temperature, at X=0, _C2=0

```
> Eq3:=subs(_C2=0,Eq2);
```

$$Eq3 := \theta_1(X) = \_C1\ \mathrm{BesselI}(0, 2\,N_1\sqrt{X})$$

```
> Eq4:=diff(Eq3,X);
```

$$Eq4 := \frac{d}{dX}\theta_1(X) = \frac{\_C1\ \mathrm{BesselI}(1, 2\,N_1\sqrt{X})\,N_1}{\sqrt{X}}$$

Next we create the differential equation for the rectangular fin and solve it.

```
> Eq5:=diff(theta[2](X),X,X)-N[2]^2*theta[2](X)=0;
```

$$Eq5 := \left(\frac{d^2}{dX^2}\theta_2(X)\right)-N_2^{\,2}\,\theta_2(X)=0$$

```
> Eq6:=dsolve(Eq5,theta[2](X));
```

$$Eq6 := \theta_2(X) = \_C1\ \mathbf{e}^{(-N_2\,X)} + \_C2\ \mathbf{e}^{(N_2\,X)}$$

```
> Eq7:=convert(Eq6,trig);
```

$$Eq7 := \theta_2(X) = \_C1\ (\cosh(N_2\,X)-\sinh(N_2\,X)) + \_C2\ (\cosh(N_2\,X)+\sinh(N_2\,X))$$

Because _C1 and _C2 appeared in the equation for θ_1, we replace the constants in Eq7 by _C3 and _C4 to avoid confusion.

```
> Eq8:=subs(_C1=_C3, _C2=_C4,Eq7);
```

$$Eq8 := \theta_2(X) = \_C3\ (\cosh(N_2\,X)-\sinh(N_2\,X)) + \_C4\ (\cosh(N_2\,X)+\sinh(N_2\,X))$$

```
> Eq9:=diff(Eq8,X);
```

$$Eq9 := \frac{d}{dX}\theta_2(X) = \_C3\,(\sinh(N_2\,X)\,N_2 - \cosh(N_2\,X)\,N_2) + \_C4\,(\sinh(N_2\,X)\,N_2 + \cosh(N_2\,X)\,N_2)$$

```
> bc1:=subs(X=f,rhs(Eq3))-subs(X=f,rhs(Eq8));
```

$$bc1 := \_C1\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{f}) - \_C3\,(\cosh(N_2\,f) - \sinh(N_2\,f)) - \_C4\,(\cosh(N_2\,f) + \sinh(N_2\,f))$$

```
> bc2:=subs(X=f,rhs(Eq4))-subs(X=f,rhs(Eq9));
```

$$bc2 := \frac{\_C1\,\mathrm{BesselI}(1, 2\,N_1\,\sqrt{f})\,N_1}{\sqrt{f}} - \_C3\,(\sinh(N_2\,f)\,N_2 - \cosh(N_2\,f)\,N_2) - \_C4\,(\sinh(N_2\,f)\,N_2 + \cosh(N_2\,f)\,N_2)$$

```
> bc3:=subs(X=1,rhs(Eq8))-1;
```

$$bc3 := \_C3\,(\cosh(N_2) - \sinh(N_2)) + \_C4\,(\cosh(N_2) + \sinh(N_2)) - 1$$

```
> const:=convert(solve({bc1=0,bc2=0,bc3=0},{_C1,_C3,_C4}),
trig):
> assign(const):
> simplify(Eq3);
```

$$\theta_1(X) = -N_2\,\sqrt{f}\,(\cosh(N_2\,f + N_2) + \sinh(N_2\,f + N_2))\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{X}) \Big/ \Big(-\mathrm{BesselI}(1, 2\,N_1\,\sqrt{f})\,N_1\,\cosh(N_2)^2 - \mathrm{BesselI}(1, 2\,N_1\,\sqrt{f})\,N_1\,\cosh(N_2)\,\sinh(N_2) + \mathrm{BesselI}(1, 2\,N_1\,\sqrt{f})\,N_1\,\cosh(N_2\,f)^2 + \mathrm{BesselI}(1, 2\,N_1\,\sqrt{f})\,N_1\,\sinh(N_2\,f)\,\cosh(N_2\,f) - N_2\,\sqrt{f}\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{f})\,\cosh(N_2)^2 - N_2\,\sqrt{f}\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{f})\,\cosh(N_2)\,\sinh(N_2) + N_2\,\sqrt{f}\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{f}) - N_2\,\sqrt{f}\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{f})\,\cosh(N_2\,f)^2 - N_2\,\sqrt{f}\,\mathrm{BesselI}(0, 2\,N_1\,\sqrt{f})\,\sinh(N_2\,f)\,\cosh(N_2\,f) \Big)$$

```
> simplify(Eq8);
```

$$\theta_2(X) = -(\cosh(N_2)+\sinh(N_2))\,(-\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\cosh(N_2 f)^2\cosh(N_2 X)$$
$$+\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\cosh(N_2 f)^2\sinh(N_2 X)$$
$$+N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\cosh(N_2 f)^2\cosh(N_2 X)$$
$$-N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\cosh(N_2 f)^2\sinh(N_2 X)$$
$$-\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\sinh(N_2 f)\cosh(N_2 f)\cosh(N_2 X)$$
$$+\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\sinh(N_2 f)\cosh(N_2 f)\sinh(N_2 X)$$
$$+N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\sinh(N_2 f)\cosh(N_2 f)\cosh(N_2 X)$$
$$-N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\sinh(N_2 f)\cosh(N_2 f)\sinh(N_2 X)$$
$$+\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\cosh(N_2 X)+N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\sinh(N_2 X))\bigg/($$
$$-\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\cosh(N_2)^2-\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\cosh(N_2)\sinh(N_2)$$
$$+\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\cosh(N_2 f)^2$$
$$+\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\sinh(N_2 f)\cosh(N_2 f)$$
$$-N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\cosh(N_2)^2$$
$$-N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\cosh(N_2)\sinh(N_2)+N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})$$
$$-N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\cosh(N_2 f)^2$$
$$-N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f})\sinh(N_2 f)\cosh(N_2 f))$$

```
> q:=k*w*((T[b]-T[a])/L)*subs(X=1,rhs(Eq9));
```

$$q := k\,w\,(T_b - T_a)\,(-(\cosh(N_2)+\sinh(N_2))\,(\cosh(N_2 f)+\sinh(N_2 f))^2$$
$$(\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1 - N_2\sqrt{f}\,\text{BesselI}(0, 2\,N_1\sqrt{f}))\,(\sinh(N_2)\,N_2-\cosh(N_2)\,N_2)$$
$$\bigg/\,(\text{BesselI}(1, 2\,N_1\sqrt{f})\,N_1\,(\cosh(N_2)+\sinh(N_2))^2$$

$$- \mathrm{BesselI}(1, 2\, N_1 \sqrt{f})\, N_1\, (\cosh(N_2 f) + \sinh(N_2 f))^2$$
$$+ N_2 \sqrt{f}\, (\cosh(N_2) + \sinh(N_2))^2\, \mathrm{BesselI}(0, 2\, N_1 \sqrt{f})$$
$$+ N_2 \sqrt{f}\, (\cosh(N_2 f) + \sinh(N_2 f))^2\, \mathrm{BesselI}(0, 2\, N_1 \sqrt{f})) + (\cosh(N_2) + \sinh(N_2))$$
$$(\mathrm{BesselI}(1, 2\, N_1 \sqrt{f})\, N_1 + N_2 \sqrt{f}\, \mathrm{BesselI}(0, 2\, N_1 \sqrt{f}))\, (\sinh(N_2)\, N_2 + \cosh(N_2)\, N_2)$$
$$\Big/ (\mathrm{BesselI}(1, 2\, N_1 \sqrt{f})\, N_1\, (\cosh(N_2) + \sinh(N_2))^2$$
$$- \mathrm{BesselI}(1, 2\, N_1 \sqrt{f})\, N_1\, (\cosh(N_2 f) + \sinh(N_2 f))^2$$
$$+ N_2 \sqrt{f}\, (\cosh(N_2) + \sinh(N_2))^2\, \mathrm{BesselI}(0, 2\, N_1 \sqrt{f})$$
$$+ N_2 \sqrt{f}\, (\cosh(N_2 f) + \sinh(N_2 f))^2\, \mathrm{BesselI}(0, 2\, N_1 \sqrt{f})))/L$$

The interface temperature, θ_i can be found by substituting X=f into either Eq. 3 or Eq. 8.
To use the above results, consider the following data. The system is assumed to have a unit depth.

```
>k:=100;w:=0.005;T[b]:=100;T[a]:=30;f:=0.5;N[2]:=1;N[1]:=N[2]*f;L:=0.05;
```

$$k := 100$$
$$w := 0.005$$
$$T_b := 100$$
$$T_a := 30$$
$$f := 0.5$$
$$N_2 := 1$$
$$N_1 := 0.5$$
$$L := 0.05$$

```
> Temp_Distribution_tirangular_fin:=simplify(Eq3);
```

$$Temp\_Distribution\_tirangular\_fin := 0.7997552669 = 0.7083994197\ \mathrm{BesselI}(0., \sqrt{X})$$

```
> Temp_Dist_Rectangular_fin:=simplify(Eq8);
```

$$Temp\_Dist\_Rectangular\_fin :=$$
$$\theta_2(X) = 0.8036496086\ \cosh(X) - 0.2043021652\ \sinh(X)$$

```
> Temp_at_Interface:=evalf(theta[i]);
```

$$Temp\_at\_Interface := 0.7997552670$$

```
> Heat_transfer_rate:=evalf(q,4);
```

$$Heat\_transfer\_rate := 440.3$$

6.6.2 Capped Hollow Tube Transistor Heat Sink

To reduce the temperature of a disk shaped transistor, it has been proposed to attach a capped hollow tube of high thermal conductivity [4]. The proposed design is shown in Fig 6.11. The steady state temperature of the transistor is T_b and heat is conducted up the hollow tube to its top and then conducted radically towards the center of the cap. Only the outside surfaces of the tube and cap are assumed to be convectively active, losing heat to the surroundings at T_a. The heat transfer coefficient for the tube surface is h_1 and that for the top surface of the cap is h_2. The tube has a length, L and a thickness, w. The top has a thickness w and

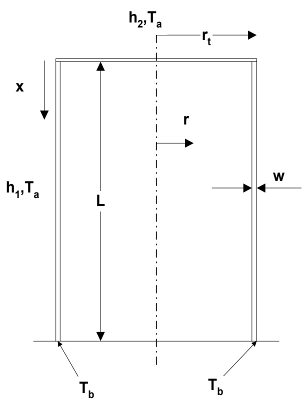


Fig. 6.11 . Capped Hollow Tube Transistor Heat Sink

The hollow tube can be treated as a straight rectangular fin with perimeter, $P = 2\pi r_t$ and a cross sectional area $A_c = P\,w$, giving $hP/kA_c = h_1/kw$. Denoting h_1/kw by m_1^2 and $(T-T_a)$ by θ, the equation governing the temperature distribution in the hollow tube becomes

$$\frac{d^2\theta(x)}{dx^2} - m_1^2\theta(x) = 0 \tag{6.44}$$

For the cap, A_c = 2 π r w and, because only one surface is convectively active, $A_s = \pi r^2$. Denoting h_2 /k w by m_2^2 and (T-T_a) by θ, the equation governing the temperature distribution in the cap becomes

$$r\frac{d^2\theta(r)}{dr^2} + \frac{d\theta(r)}{dr} - m_2^2\theta(r) = 0 \tag{6.45}$$

The four boundary conditions are

$$\theta(x = L) - \theta_b = 0 \tag{6.46a}$$

$$\theta(x = 0) - \theta(r = r_t) = 0 \tag{6.46b}$$

$$\frac{d\theta(x)}{dx}(x = 0) = \frac{d\theta(r)}{dr}(r = r_t) \tag{6.46c}$$

and

$$\theta(r = 0) = finite \tag{6.46d}$$

The heat transfer rate is given by

$$q = k(2\pi r_t w)\left.\frac{d\theta(x)}{dx}\right|_{x=L} \tag{6.47}$$

Example 6.13
We first create the differential equation for the temperature distribution in the hollow tube (straight fin).

```
> restart;
> Eq1:=diff(theta(x),x,x)-m[1]^2*theta(x)=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\theta(x)\right) - m_1^{\,2}\,\theta(x) = 0$$

```
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\, e^{(m_1 x)} + \_C2\, e^{(-m_1 x)}$$

```
> Eq3:=convert(Eq2,trig);
```

$$Eq3 := \theta(x) = \_C1\,(\cosh(m_1 x) + \sinh(m_1 x)) + \_C2\,(\cosh(m_1 x) - \sinh(m_1 x))$$

```
> Eq4:=diff(Eq3,x);
```

$$Eq4 := \frac{d}{dx}\theta(x) = \_C1\,(\sinh(m_1 x)\, m_1 + \cosh(m_1 x)\, m_1) + \_C2\,(\sinh(m_1 x)\, m_1 - \cosh(m_1 x)\, m_1)$$

Next we create the differential equation for the temperature distribution in the cap.

```
> Eq5:=r*diff(theta(r),r,r)+diff(theta(r),r)-
m[2]^2*r*theta(r)=0;
```

$$Eq5 := r\left(\frac{d^2}{dr^2}\theta(r)\right) + \left(\frac{d}{dr}\theta(r)\right) - {m_2}^2\, r\, \theta(r) = 0$$

```
> Eq6:=dsolve(Eq5,theta(r));
```

$$Eq6 := \theta(r) = \_C1\ \mathrm{BesselI}(0, m_2\, r) + \_C2\ \mathrm{BesselK}(0, m_2\, r)$$

```
> Eq7:=subs(_C1=_C3,_C2=_C4,Eq6);
```

$$Eq7 := \theta(r) = \_C3\ \mathrm{BesselI}(0, m_2\, r) + \_C4\ \mathrm{BesselK}(0, m_2\, r)$$

For finite temperature at r = 0, _C4 must equal 0.

```
> _C4:=0;
```

$$\_C4 := 0$$

```
> Eq8:=diff(Eq7,r);
```

$$Eq8 := \frac{d}{dr}\theta(r) = \_C3\ \mathrm{BesselI}(1, m_2\, r)\, m_2$$

Now we write the boundary conditions and obtain the temperature distributions in the tube and the cap. Finally we find the rate of heat transfer.

```
> bc1:=simplify(subs(x=0,rhs(Eq3))-subs(r=r[t],rhs(Eq7)));
```

$$bc1 := \_C1 + \_C2 - \_C3\ \mathrm{BesselI}(0, m_2\, r_t)$$

```
> bc2:=simplify(subs(x=0,rhs(Eq4))-subs(r=r[t],rhs(Eq8)));
```

$$bc2 := \_C1\ m_1 - \_C2\ m_1 - \_C3\ \mathrm{BesselI}(1, m_2\, r_t)\, m_2$$

```
> bc3:=simplify(subs(x=L,rhs(Eq3))-theta[b]);
```

$$bc3 := \_C1\ \cosh(m_1\, L) + \_C1\ \sinh(m_1\, L) + \_C2\ \cosh(m_1\, L) - \_C2\ \sinh(m_1\, L) - \theta_b$$

```
> consts:=solve({bc1=0,bc2=0,bc3=0},{_C1,_C2,_C3}):
> assign(consts):
> Eq9:=simplify(convert(Eq3,trig));
```

$$Eq9 := \theta(x) = \theta_b\,(\cosh(m_1\, L) + \sinh(m_1\, L))\,(m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, x) + \mathrm{BesselI}(1, m_2\, r_t)\, m_2 \sinh(m_1\, x)) \Big/ (m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, L)^2 + m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, L)\sinh(m_1\, L) + \mathrm{BesselI}(1, m_2\, r_t)\, m_2\cosh(m_1\, L)^2 + \mathrm{BesselI}(1, m_2\, r_t)\, m_2 \cosh(m_1\, L)\sinh(m_1\, L) - \mathrm{BesselI}(1, m_2\, r_t)\, m_2)$$

```
> Eq10:=simplify(convert(Eq7,trig));
```

$$Eq10 := \theta(r) = \theta_b\,(\cosh(m_1\, L) + \sinh(m_1\, L))\, m_1\,\mathrm{BesselI}(0, m_2\, r) \Big/ (m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, L)^2 + m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, L)\sinh(m_1\, L) + \mathrm{BesselI}(1, m_2\, r_t)\, m_2\cosh(m_1\, L)^2 + \mathrm{BesselI}(1, m_2\, r_t)\, m_2 \cosh(m_1\, L)\sinh(m_1\, L) - \mathrm{BesselI}(1, m_2\, r_t)\, m_2)$$

```
> q:=simplify(k*w*2*Pi*r[t]*convert(subs(x=L,rhs(Eq4)),
trig));
```

$$q := 2\, k\, w\, \pi\, r_t\, m_1\, \theta_b\,(\cosh(m_1\, L) + \sinh(m_1\, L))\,(\mathrm{BesselI}(1, m_2\, r_t)\, m_2\cosh(m_1\, L) + m_1\,\mathrm{BesselI}(0, m_2\, r_t)\sinh(m_1\, L)) \Big/ (m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, L)^2 + m_1\,\mathrm{BesselI}(0, m_2\, r_t)\cosh(m_1\, L)\sinh(m_1\, L) + \mathrm{BesselI}(1, m_2\, r_t)\, m_2\cosh(m_1\, L)^2 + \mathrm{BesselI}(1, m_2\, r_t)\, m_2 \cosh(m_1\, L)\sinh(m_1\, L) - \mathrm{BesselI}(1, m_2\, r_t)\, m_2)$$

The special case of identical heat transfer coefficients for the tube surface and for the cap surface can be derived by setting $h_1 = h_2$ which then gives $m_1 = m_2 = m$.

```
> Eq11:=simplify(subs(m[1]=m,m[2]=m,Eq9));
```

$$Eq11 := \theta(x) = \theta_b\,(\cosh(m\,L) + \sinh(m\,L))\,(\mathrm{BesselI}(0, m\,r_t)\cosh(m\,x) + \mathrm{BesselI}(1, m\,r_t)\sinh(m\,x)) \Big/ (\mathrm{BesselI}(0, m\,r_t)\cosh(m\,L)^2 + \mathrm{BesselI}(0, m\,r_t)\cosh(m\,L)\sinh(m\,L) + \mathrm{BesselI}(1, m\,r_t)\cosh(m\,L)^2 + \mathrm{BesselI}(1, m\,r_t)\cosh(m\,L)\sinh(m\,L) - \mathrm{BesselI}(1, m\,r_t))$$

```
> Eq12:=simplify(subs(m[1]=m,m[2]=m,Eq10));
```

$$Eq12 := \theta(r) = \theta_b\,(\cosh(m\,L) + \sinh(m\,L))\,\mathrm{BesselI}(0, m\,r) \Big/ (\mathrm{BesselI}(0, m\,r_t)\cosh(m\,L)^2 + \mathrm{BesselI}(0, m\,r_t)\cosh(m\,L)\sinh(m\,L) + \mathrm{BesselI}(1, m\,r_t)\cosh(m\,L)^2 + \mathrm{BesselI}(1, m\,r_t)\cosh(m\,L)\sinh(m\,L) - \mathrm{BesselI}(1, m\,r_t))$$

```
> q_special_case:=simplify(subs(m[1]=m,m[2]=m,q));
```

$$q\_special\_case := 2\,k\,w\,\pi\,r_t\,m\,\theta_b\,(\cosh(m\,L) + \sinh(m\,L))\,(\mathrm{BesselI}(1, m\,r_t)\cosh(m\,L) + \mathrm{BesselI}(0, m\,r_t)\sinh(m\,L)) \Big/ (\mathrm{BesselI}(0, m\,r_t)\cosh(m\,L)^2 + \mathrm{BesselI}(0, m\,r_t)\cosh(m\,L)\sinh(m\,L) + \mathrm{BesselI}(1, m\,r_t)\cosh(m\,L)^2 + \mathrm{BesselI}(1, m\,r_t)\cosh(m\,L)\sinh(m\,L) - \mathrm{BesselI}(1, m\,r_t))$$

We use the above analysis to calculate the heat loss, temperature at tube-cap junction, and temperature at the center of the cap. The parameters used pertain to a typical system design.

```
>k:=25;h:=300;w:=0.0003;m:=sqrt(h/(k*w));L:=0.009;r[t]:=0.004;T[a]:=20;T[b]:=100;theta[b]:=T[b]-T[a];
```

$$k := 25$$
$$h := 300$$
$$w := 0.0003$$
$$m := 200.0000000$$
$$L := 0.009$$

$$r_t := 0.004$$
$$T_a := 20$$
$$T_b := 100$$
$$\theta_b := 80$$

```
> evalf(q_special_case,4);
```

2.942

```
> T[j]:=T[a]+evalf(subs(x=0,rhs(Eq11)),4);
```

$$T_j := 39.05$$

```
> T[c]:=T[a]+evalf(subs(r=0,rhs(Eq12)),4);
```

$$T_c := 36.32$$

6.6.3 The Y-Shaped Fin Array

We now consider a Y-shaped array consisting of two triangular fins and a rectangular fin as shown in Fig 6.12 This configuration has been recently analyzed by Aziz and McFadden [5]. The thickness of the rectangular fin is w which is also the base thickness of the two triangular fins The fin lengths are L_1, L_2 , and L_3. Although all three fins operate in a common convective environment at T_a, each fin has its own heat transfer coefficient. These are identified as h_1, h_2, and h_3. All three fins are made of the same material having thermal conductivity, k.

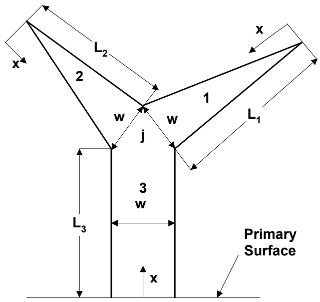


Fig 6.12: Y-shaped fin arra

With $\theta = T - T_a$, the differential equations for the individual fins may be written as

Fin 1:
$$x\frac{d^2\theta_1(x)}{dx^2}+\frac{d\theta_1(x)}{dx}-m_1{}^2\theta_1(x)=0 \tag{6.48}$$

Fin 2:
$$x\frac{d^2\theta_2(x)}{dx^2}+\frac{d\theta_2(x)}{dx}-m_2{}^2\theta_2(x)=0 \tag{6.49}$$

Fin 3:
$$\frac{d^2\theta_3(x)}{dx^2}-m_2{}^2\theta_3(x)=0 \tag{6.50}$$

where $m_1^2=2h_1/kw,\quad m_2^2=2h_2/kw,\quad m_3^2=2h_3/kw$

Three boundary conditions are established by specifying the heat inputs q_1,q_2 and q_3 at the free ends of the fins. The requirement that each fin must have the same temperature, θ_j, at the junction of the three fins gives three other boundary conditions. The unknown junction temperature, θ_j, can be found by invoking the heat flux balance (continuity) at the junction. For a depth of H, the six boundary conditions are

$$kwH\left.\frac{d\theta_1}{dx_1}\right|_{x_1=0}+q_1=0 \tag{6.51}$$

$$kwH\left.\frac{d\theta_2}{dx_2}\right|_{x_2=0}+q_2=0 \tag{6.52}$$

$$kwH\left.\frac{d\theta_3}{dx_3}\right|_{x_3=0}+q_3=0 \tag{6.53}$$

$$\theta_1(x_1=L_1)-\theta_j=0 \tag{6.54}$$

$$\theta_2(x_2=L_2)-\theta_j=0 \tag{6.55}$$

$$\theta_3(x_3=L_3)-\theta_j=0 \tag{6.56}$$

and the heat flux balance at the junction gives

$$\frac{d\theta_1}{dx_1}(x_1 = L_1) + \frac{d\theta_2}{dx_2}(x_2 = L_2) + \frac{d\theta_3}{dx_3}(x_3 = L_3) = 0 \tag{6.57}$$

Example 6.14

```
> restart;
> Eq1:=diff(theta[1](x),x,x)-m[1]^2*theta[1](x)=0;
```

$$Eq1 := \left(\frac{d^2}{dx^2}\theta_1(x)\right) - m_1^{\ 2}\,\theta_1(x) = 0$$

```
> Eq2:=dsolve(Eq1,theta[1](x));
```

$$Eq2 := \theta_1(x) = \_C1\, e^{(-m_1 x)} + \_C2\, e^{(m_1 x)}$$

```
> Eq3:=convert(Eq2,trig);
```

$$Eq3 := \theta_1(x) = \_C1\,(\cosh(m_1 x) - \sinh(m_1 x)) + \_C2\,(\cosh(m_1 x) + \sinh(m_1 x))$$

```
> Eq4:=diff(Eq3,x);
```

$$Eq4 := \frac{d}{dx}\theta_1(x) =$$
$$\_C1\,(\sinh(m_1 x)\, m_1 - \cosh(m_1 x)\, m_1) + \_C2\,(\sinh(m_1 x)\, m_1 + \cosh(m_1 x)\, m_1)$$

```
> Eq5:=diff(theta[2](x),x,x)-m[2]^2*theta[2](x)=0;
```

$$Eq5 := \left(\frac{d^2}{dx^2}\theta_2(x)\right) - m_2^{\ 2}\,\theta_2(x) = 0$$

```
> Eq6:=dsolve(Eq5,theta[2](x));
```

$$Eq6 := \theta_2(x) = \_C1\, e^{(m_2 x)} + \_C2\, e^{(-m_2 x)}$$

```
> Eq7:=subs(_C1=_C3,_C2=_C4,Eq6);
```

$$Eq7 := \theta_2(x) = \_C3\, e^{(m_2 x)} + \_C4\, e^{(-m_2 x)}$$

```
> Eq8:=convert(Eq7,trig);
```

$$Eq8 := \theta_2(x) = \_C3\ (\cosh(m_2\,x) + \sinh(m_2\,x)) + \_C4\ (\cosh(m_2\,x) - \sinh(m_2\,x))$$

```
> Eq9:=diff(Eq8,x);
```

$$Eq9 := \frac{d}{dx}\theta_2(x) = \_C3\ (\sinh(m_2\,x)\,m_2 + \cosh(m_2\,x)\,m_2) + \_C4\ (\sinh(m_2\,x)\,m_2 - \cosh(m_2\,x)\,m_2)$$

```
> Eq10:=diff(theta[3](x),x,x)-m[3]^2*theta[3](x)=0;
```

$$Eq10 := \left(\frac{d^2}{dx^2}\theta_3(x)\right) - m_3^{\,2}\,\theta_3(x) = 0$$

```
> Eq11:=dsolve(Eq10,theta[3](x));
```

$$Eq11 := \theta_3(x) = \_C1\ \mathrm{e}^{(-m_3\,x)} + \_C2\ \mathrm{e}^{(m_3\,x)}$$

```
> Eq12:=subs(_C1=_C5,_C2=_C6,Eq11);
```

$$Eq12 := \theta_3(x) = \_C5\ \mathrm{e}^{(-m_3\,x)} + \_C6\ \mathrm{e}^{(m_3\,x)}$$

```
> Eq13:=convert(Eq12,trig);
```

$$Eq13 := \theta_3(x) = \_C5\ (\cosh(m_3\,x) - \sinh(m_3\,x)) + \_C6\ (\cosh(m_3\,x) + \sinh(m_3\,x))$$

```
> Eq14:=diff(Eq13,x);
```

$$Eq14 := \frac{d}{dx}\theta_3(x) = \_C5\ (\sinh(m_3\,x)\,m_3 - \cosh(m_3\,x)\,m_3) + \_C6\ (\sinh(m_3\,x)\,m_3 + \cosh(m_3\,x)\,m_3)$$

Now we generate the specified heat input boundary conditions.

```
> bc1:=simplify(k*H*w*evalf(subs(x=0,rhs(Eq4))))+q[1];
```

$$bc1 := -1.\ k\,H\,w\,m_1\ (\_C1 - 1.\ \_C2) + q_1$$

```
> bc2:=simplify(k*H*w*evalf(subs(x=0,rhs(Eq9))))+q[2];
```

$$bc2 := k\,H\,w\,m_2\ (\_C3 - 1.\ \_C4) + q_2$$

```
> bc3:=simplify(k*H*w*evalf(subs(x=0,rhs(Eq14))))+q[3];
```

$$bc3 := -1.\ k\ H\ w\ m_3\ (\_C5 - 1.\ \_C6) + q_3$$

Finally we generate three boundary conditions by specifying the common junction temperature.

```
> bc4:=subs(x=L[1],rhs(Eq3))-theta[j];
```

$$bc4 := \_C1\ (\cosh(m_1\ L_1) - \sinh(m_1\ L_1)) + \_C2\ (\cosh(m_1\ L_1) + \sinh(m_1\ L_1)) - \theta_j$$

```
> bc5:=subs(x=L[2],rhs(Eq8))-theta[j];
```

$$bc5 := \_C3\ (\cosh(m_2\ L_2) + \sinh(m_2\ L_2)) + \_C4\ (\cosh(m_2\ L_2) - \sinh(m_2\ L_2)) - \theta_j$$

```
> bc6:=subs(x=L[3],rhs(Eq13))-theta[j];
```

$$bc6 := \_C5\ (\cosh(m_3\ L_3) - \sinh(m_3\ L_3)) + \_C6\ (\cosh(m_3\ L_3) + \sinh(m_3\ L_3)) - \theta_j$$

```
> consts:=solve({bc1=0,bc2=0,bc3=0,bc4=0,bc5=0,bc6=0},{_C1,
_C2,_C3,_C4,_C5,_C6}):
> assign(consts):
> Eq15:=simplify(convert(Eq3,trig));
```

$$Eq15 := \theta_1(x) = (q_1 \cosh(m_1\ L_1)^2 \cosh(m_1\ x) - 1.\ q_1 \cosh(m_1\ L_1)^2 \sinh(m_1\ x) + \cosh(m_1\ L_1)\ q_1 \sinh(m_1\ L_1) \cosh(m_1\ x) - 1.\ \cosh(m_1\ L_1)\ q_1 \sinh(m_1\ L_1) \sinh(m_1\ x) + \theta_j\ k\ H\ w\ m_1 \cosh(m_1\ L_1) \cosh(m_1\ x) - 1.\ q_1 \cosh(m_1\ x) + \theta_j\ k\ H\ w\ m_1 \sinh(m_1\ L_1) \cosh(m_1\ x)) \Big/ (k\ H\ w\ m_1 \cosh(m_1\ L_1)\ (\cosh(m_1\ L_1) + \sinh(m_1\ L_1)))$$

```
> Eq16:=simplify(convert(Eq8,trig));
```

$$Eq16 := \theta_2(x) = (\theta_j\ k\ H\ w\ m_2 \cosh(m_2\ L_2) \cosh(m_2\ x) + \theta_j\ k\ H\ w\ m_2 \sinh(m_2\ L_2) \cosh(m_2\ x) - 1.\ q_2 \cosh(m_2\ x) + q_2 \cosh(m_2\ L_2)^2 \cosh(m_2\ x) - 1.\ q_2 \cosh(m_2\ L_2)^2 \sinh(m_2\ x) + \cosh(m_2\ L_2)\ q_2 \sinh(m_2\ L_2) \cosh(m_2\ x) - 1.\ \cosh(m_2\ L_2)\ q_2 \sinh(m_2\ L_2) \sinh(m_2\ x)) \Big/ (k\ H\ w\ m_2 \cosh(m_2\ L_2)\ (\cosh(m_2\ L_2) + \sinh(m_2\ L_2)))$$

```
> Eq17:=simplify(convert(Eq13,trig));
```

$$Eq17 := \theta_3(x) = (q_3 \cosh(m_3 L_3)^2 \cosh(m_3 x) - 1.\ q_3 \cosh(m_3 L_3)^2 \sinh(m_3 x) + \cosh(m_3 L_3)\, q_3 \sinh(m_3 L_3) \cosh(m_3 x) - 1.\ \cosh(m_3 L_3)\, q_3 \sinh(m_3 L_3) \sinh(m_3 x) + \theta_j\, k\, H\, w\, m_3 \cosh(m_3 L_3) \cosh(m_3 x) - 1.\ q_3 \cosh(m_3 x) + \theta_j\, k\, H\, w\, m_3 \sinh(m_3 L_3) \cosh(m_3 x)) \Big/ (k\, H\, w\, m_3 \cosh(m_3 L_3)\,(\cosh(m_3 L_3) + \sinh(m_3 L_3)))$$

Because the sum of the heat fluxes at the junction must be zero, we apply this condition to derive an expression for θ_j.

```
> Eq18:=subs(x=L[1],rhs(Eq4))+subs(x=L[2],rhs(Eq9))+
subs(x=L[3],rhs(Eq14)):
```

```
> theta[j]:=simplify(solve(Eq18=0,theta[j])):
```

Let us consider the specified example with the following data.

```
> L[1]:=0.04;L[2]:=L[1];L[3]:=L[1];w:=0.0025;k:=200;h:=100;
q[1]:=40;q[2]:=80;q[3]:=20;m[1]:=sqrt(2*h/(k*w));m[2]:=m[1];
m[3]:=m[1];H:=0.2;
```

$$L_1 := 0.04$$
$$L_2 := 0.04$$
$$L_3 := 0.04$$
$$w := 0.0025$$
$$k := 200$$
$$h := 100$$
$$q_1 := 40$$
$$q_2 := 80$$
$$q_3 := 20$$
$$m_1 := 20.00000000$$
$$m_2 := 20.00000000$$
$$m_3 := 20.00000000$$
$$H := 0.2$$

The results match those that can be obtained using the method of nodal analysis.

```
> evalf(subs(x=0,Eq15),4);
```

$$\theta_1(0) = 32.90$$

```
> evalf(subs(x=0,Eq16),4);
```

$$\theta_2(0) = 46.18$$

```
> evalf(subs(x=0,Eq17),4);
```

$$\theta_3(0) = 26.26$$

```
> Junct_temp:=evalf(theta[j],4);
```

$$Junct\_temp := 26.22$$

6.7 CONVECTING-RADIATING FINS

In some fin applications, the surface radiation becomes important and must be included in the energy balance used for writing the fin differential equation for the temperature distribution. For example, when a tube-sheet radiator operates in space, the convection environment is absent and the only mechanism available for heat dissipation from the sheet (fin) surface is radiation to the deep space. Likewise, when a heat sink is cooled by natural convection, radiative heat loss may be comparable to convective heat loss, particularly at moderate or high operating temperatures.

To study the effects of radiative heat loss from a fin, we consider a straight rectangular fin of section 6.3.1 (Fig 6.2). Let ε be the emissivity of the fin surface and let the fin radiate to large surroundings at a temperature of T_s. With the inclusion of the radiation heat loss term, Eq. (6.3) becomes

$$\frac{d^2T}{dx^2} - \frac{hP}{kA_c}(T - T_s) - \frac{\varepsilon\sigma(T^4 - T_s^4)}{kA_c} = 0 \qquad (6.57)$$

With the introduction of several dimensionless quantities

$$\theta = \frac{T}{T_b}, \quad X = \frac{x}{L}, \quad N_c = \frac{hPL^2}{kA_c}, \quad N_r = \frac{\varepsilon\sigma PT_b^3}{kA_c} \qquad (6.59)$$

and noting that x is measured from the tip of the fin, equation 6.58 becomes

$$\frac{d^2\theta}{dX^2} - N_c(\theta - \theta_a) - N_r(\theta^4 - \theta_s^4) = 0 \qquad (6.60)$$

For a constant base temperature, T_b, and an insulated tip, the boundary conditions may be written as

$$\left.\frac{d\theta}{dX}\right|_{X=0} = 0 \quad and \quad \theta(X=1) = 1 \tag{6.61}$$

The heat transfer from the fin is given by

$$q = kw\left.\frac{dT}{dx}\right|_{x=L} \tag{6.62}$$

or in dimensionless form by

$$\frac{qL}{kwT_b} = \left.\frac{d\theta}{dX}\right|_{X=1} \tag{6.63}$$

In Example 6.15, Eq. (6.60) is solved subject to the boundary conditions in Eq. (6.61). Because Eq. (6.60) does not admit an exact analytical solution, the numeric option must be used to obtain the solution. For this purpose, we take $N_c = N_r = 1$ and adopt a shooting technique in which a succession of guessed values for θ(0) are used to solve Eq. (6.60). The sequence of solutions is examined to see if the boundary condition of θ(0)=1 is met. A scheme of linear interpolation is used to refine the guess for θ(0) until θ(1) = 1 is met within a certain tolerance.

Example 6.15

```
> restart;
> Eq1:=diff(theta(X),X,X)-theta(X)-theta(X)^4=0;
```

$$Eq1 := \left(\frac{d^2}{dX^2}\theta(X)\right) - \theta(X) - \theta(X)^4 = 0$$

```
> guesses:=[0.1,0.2,0.3,0.4,0.5,0.6,0.7,0.8];
```

$$guesses := [0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8]$$

```
> approx:=proc(guess)
> dsolve({Eq1,theta(0)=guess,D(theta)(0)=0},theta(X),
numeric);
> end:
```

```
> Eq2:=map(approx,guesses);
```

$$Eq2 := [\text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}, \text{proc}(x\_rkf45) \ \ldots \ \text{end proc}]$$

```
> for k from 1 to 6 by 1 do print(k,Eq2[k](1));od;
```

$$1, \left[X = 1., \theta(X) = 0.154386885180970657, \frac{d}{dX}\theta(X) = 0.117755684034573468\right]$$

$$2, \left[X = 1., \theta(X) = 0.309882194247591224, \frac{d}{dX}\theta(X) = 0.238834736663227604\right]$$

$$3, \left[X = 1., \theta(X) = 0.469394355599112300, \frac{d}{dX}\theta(X) = 0.372120747162978161\right]$$

$$4, \left[X = 1., \theta(X) = 0.638066547760878790, \frac{d}{dX}\theta(X) = 0.534171426323040932\right]$$

$$5, \left[X = 1., \theta(X) = 0.824051017692750243, \frac{d}{dX}\theta(X) = 0.754025401013110108\right]$$

$$6, \left[X = 1., \theta(X) = 1.04033756836731772, \frac{d}{dX}\theta(X) = 1.08565429805617942\right]$$

The above sequence of solutions indicate that the correct solution (θ(1)=1) lies between the last two solutions. To guess the value of θ(0) we use linear interpolation as shown below.

```
> x:=[0.824051,1.040337];
```

$$x := [0.824051, 1.040337]$$

```
> y:=[0.5,0.6];
```

$$y := [0.5, 0.6]$$

```
> interp(x,y,z);
```

$$0.4623507763 \ z + 0.1189993804$$

```
> refined_guess1:=subs(z=1,%);
```

$$refined\_guess1 := 0.5813501567$$

```
> Eq3:=dsolve({Eq1,theta(0)=refined_guess1,D(theta)(0)=0},
theta(X),numeric);
```

$$Eq3 := \textbf{proc}\ (x\_rkf45)\ \ldots\ \textbf{end proc}$$

```
> Eq3(1);
```

$$\left[X = 1.,\ \theta(X) = 0.996865439207339944\ ,\ \frac{d}{dX}\theta(X) = 1.01142377211216017\right]$$

This solution does not quite match the boundary condition of θ(1)=1. We therefore again use linear interpolation to obtain the second refined guess.

```
> x:=[0.996865,1.040337];
```

$$x := [0.996865\ ,\ 1.040337\ ]$$

```
> y:=[0.581350,0.6];
```

$$y := [0.581350\ ,\ 0.6\ ]$$

```
> interp(x,y,z);
```

$$0.4290117778\ z + 0.1536831741$$

```
> refined_guess2:=subs(z=1,%);
```

$$refined\_guess2\ := 0.5826949519$$

```
> Eq4:=dsolve({Eq1,theta(0)=refined_guess2,D(theta)(0)=0},
theta(X),numeric);
```

$$Eq4 := \textbf{proc}\ (x\_rkf45)\ \ldots\ \textbf{end proc}$$

```
> Eq4(1);
```

$$\left[X = 1.,\ \theta(X) = 0.999944622960057704\ ,\ \frac{d}{dX}\theta(X) = 1.01654884769844056\right]$$

We are still not quite close enough, so we iterate again.

```
> x:=[0.999945,1.040337];
```

$$x := [0.999945\ ,\ 1.040337\ ]$$

```
> y:=[0.582695,0.6];
```

$$y := [0.582695\ ,\ 0.6\ ]$$

```
> interp(x,y,z);
```

$$0.4284264211\ z + 0.1542921424$$

```
> final_guess3:=subs(z=1,%);
```

$$final\_guess3 := 0.5827185635$$

```
> Eq5:=dsolve({Eq1,theta(0)=final_guess3,D(theta)(0)=0},
theta(X),numeric);
```

$$Eq5 := \textbf{proc}\ (x\_rkf45)\ \ldots\ \textbf{end proc}$$

```
> Eq5(1);
```

$$\left[X = 1.,\ \theta(X) = 0.999998761188855190\ ,\ \frac{d}{dX}\theta(X) = 1.01663913574180231\right]$$

At this stage, we terminate the iteration process. Eq. 5 gives us the solution for the temperature distribution. The value of dθ/dX at X = 1 gives us the heat transfer from the fin via Eq. (6.63).

The solution represented by Eq. 5 can be graphed using the command **odeplot** obtained by loading the plots package. The numerical solution for $0 \le X \le 1$ is then plotted as shown in Fig. 6.13. Alternatively, the numerical solution can be printed as well.

```
> with(plots);
```

```
> odeplot(Eq5,[X,theta(X)],0..1,labels=["X","theta"],
color=black);
```

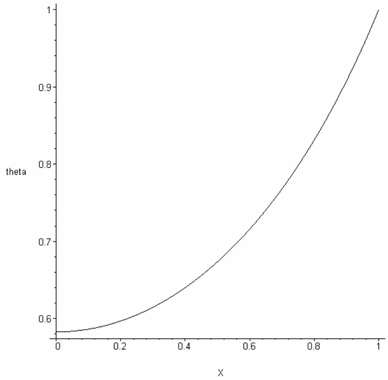


Fig 6.13: Non Dimensional temperature versus distance

```
> for k from 0 by 0.1 to 1 do print(k,Eq5(k));od;
```

$$0, \left[X = 0., \theta(X) = 0.58271856350000, \frac{d}{dX}\theta(X) = 0. \right]$$

$$0.1, \left[X = 0.1, \theta(X) = 0.586213883927524270, \frac{d}{dX}\theta(X) = 0.0700111184991064756 \right]$$

$$0.2, \left[X = 0.2, \theta(X) = 0.596763063051262254, \frac{d}{dX}\theta(X) = 0.141293566709254942 \right]$$

$$0.3, \left[X = 0.3, \theta(X) = 0.614560022197606416, \frac{d}{dX}\theta(X) = 0.215205482447867414 \right]$$

$$0.4, \left[X = 0.4, \theta(X) = 0.639943002307054698, \frac{d}{dX}\theta(X) = 0.293294135612211238 \right]$$

$$0.5, \left[X = 0.5, \theta(X) = 0.673419939651547228, \frac{d}{dX}\theta(X) = 0.377433175762529050 \right]$$

$$0.6, \left[X = 0.6, \theta(X) = 0.715710456932287408, \frac{d}{dX}\theta(X) = 0.470025088269511448 \right]$$

$$0.7, \left[X = 0.7, \theta(X) = 0.767813410404254104, \frac{d}{dX}\theta(X) = 0.574321279944406071 \right]$$

$$0.8, \left[X = 0.8, \theta(X) = 0.831116361638845502, \frac{d}{dX}\theta(X) = 0.694942243190410648 \right]$$

$$0.9, \left[X = 0.9, \theta(X) = 0.907571625109922620, \frac{d}{dX}\theta(X) = 0.838777694805622476 \right]$$

$$1.0, \left[X = 1.0, \theta(X) = 0.99999876118855190, \frac{d}{dX}\theta(X) = 1.01663913574180231 \right]$$

REFERENCES

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Chapter 7

Two Dimensional Steady Conduction

7.1 INTRODUCTION

In developing the conduction heat transfer models in the previous chapters, conduction was assumed to occur in one dimension. Such models are not adequate in many practical situations where the dimensions of the conduction region and the spatial variation of the boundary conditions on the boundary of the region induce multidimensional effects. The simplest multidimensional model is that of two dimensional steady conduction which is the subject of the present chapter. Unlike one dimensional conduction in which the heat flux lines point in a single direction, the heat flux lines in two dimensional conduction flare out in two dimensions such that the orthogonality of the isotherms and heat flux lines is preserved.

Three widely used methods for solving two dimensional steady state conduction problems are the method of separation of variables [1,2], the conduction shape factor method, and the numerical method [3]. The method of separation of variables is suitable for simple geometries and homogeneous boundary conditions and results in an exact analytic solution. The solution is typically in the form of an infinite series involving complex mathematical functions. The solution is nonetheless instructive because it not only demonstrates the elegance of mathematics, but also gives the temperature distribution as a continuous function of the two coordinates.

The conduction shape factor approach permits a quick calculation of the heat transfer rate but not the temperature distribution. The shape factors for the two dimensional configurations, which are derived using analytical methods such as conformal mapping, the superposition of heat sources and sinks, and the electrical analogy are available in heat transfer textbooks. For complex geometries and boundary conditions, the only suitable tool for analysis is a numerical procedure employing finite differences, finite elements, or boundary elements.

Because the use of the conduction shape factor entails straight forward calculations that can be performed with an ordinary calculator, it will not be discussed any further. For both the separation of variables method and the numerical approach, *Maple* provides an efficient computational tool as demonstrated in the sections that follow.

7.2 THE METHOD OF SEPARATION OF VARIABLES IN CARTESIAN COORDINATES

Consider the rectangular plate in Fig. 7.1. The plate dimensions are L and W. For a constant thermal conductivity and no internal heat generation, the two dimensional steady temperature distribution of the plate is governed by the second order partial differential equation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}=0 \tag{7.1}$$

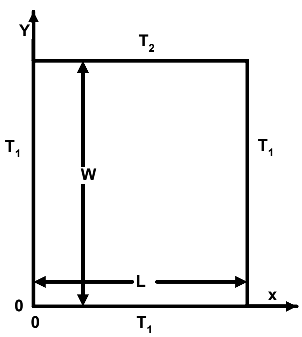


Fig 7.1 Rectangular Plate

The three sides of the plate are maintained at a constant temperature, T_1, while the fourth side is maintained at a temperature, $T_2 \neq T_1$. To simplify the solution, we introduce the transformation

$$\theta=\frac{T-T_1}{T_2-T_1} \tag{7.2}$$

which when substituted into Eq. (7.1) gives the following partial differential equation

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \tag{7.3}$$

The four boundary conditions in terms of the variable, θ, are

$$\begin{aligned}\theta(0,y)&=0\\ \theta(x,0)&=0\\ \theta(L,y)&=0\\ \theta(x,W)&=1\end{aligned} \tag{7.4}$$

Example 7.1 illustrates the solution based on the separation of variables method.

Example 7.1

```
> restart;
>Eq1:=diff(theta(x,y),x,x)+diff(theta(x,y),y,y)=;
```

$$Eq1 := \left(\frac{\partial^2}{\partial x^2}\theta(x,y)\right)+\left(\frac{\partial^2}{\partial y^2}\theta(x,y)\right)=0$$

Next a separation of variables solution is assumed with X being a function of x alone and Y being a function of y alone.

```
> theta(x,y):=X(x)*Y(y);
```

$$\theta(x,y) := X(x)\,Y(y)$$

```
> Eq2:=Eq1/theta(x,y);
```

$$Eq2 := \frac{\left(\frac{d^2}{dx^2}X(x)\right)Y(y)+X(x)\left(\frac{d^2}{dy^2}Y(y)\right)}{X(x)\,Y(y)}=0$$

```
> Eq3:=expand(Eq2);
```

$$Eq3 := \frac{\frac{d^2}{dx^2}X(x)}{X(x)}+\frac{\frac{d^2}{dy^2}Y(y)}{Y(y)}=0$$

Here we identify the separation constant to be –λ2 and add it to the left side of the resulting equation, resulting in Eq4 and Eq5, which are subsequently solved.

```
> Eq4:=expand(X(x)*(op(1,lhs(Eq3))+lambda^2));
```

$$Eq4 := \left(\frac{d^2}{dx^2}\mathrm{X}(x)\right) + \mathrm{X}(x)\,\lambda^2$$

```
> sol1:=dsolve(Eq4,X(x));
```

$$sol1 := \mathrm{X}(x) = \_C1\,\sin(\lambda\,x) + \_C2\,\cos(\lambda\,x)$$

```
> Eq5:=expand(Y(y)*(op(2,lhs(Eq3))-lambda^2));
```

$$Eq5 := \left(\frac{d^2}{dy^2}\mathrm{Y}(y)\right) - \mathrm{Y}(y)\,\lambda^2$$

```
> sol2:=dsolve(Eq5,Y(y));
```

$$sol2 := \mathrm{Y}(y) = \_C1\,\mathrm{e}^{(\lambda\,y)} + \_C2\,\mathrm{e}^{(-\lambda\,y)}$$

The constant names in sol2 are changed to prevent confusion, resulting in sol3.

```
> sol3:=subs(_C1=_C3,_C2=_C4,sol2);
```

$$sol3 := \mathrm{Y}(y) = \_C3\,\mathrm{e}^{(\lambda\,y)} + \_C4\,\mathrm{e}^{(-\lambda\,y)}$$

The solution for θ is the product of X(x) and Y(y).

```
> sol4:=rhs(sol1)*rhs(sol3);
```

$$sol4 := (\_C1\,\sin(\lambda\,x) + \_C2\,\cos(\lambda\,x))\,(\_C3\,\mathrm{e}^{(\lambda\,y)} + \_C4\,\mathrm{e}^{(-\lambda\,y)})$$

We now invoke the boundary condition θ(0,y) = 0.

```
> eval(subs(x=0,sol4))=0;
```

$$\_C2\,(\_C3\,\mathrm{e}^{(\lambda\,y)} + \_C4\,\mathrm{e}^{(-\lambda\,y)}) = 0$$

```
> _C2:=0;
```

$$\_C2 := 0$$

```
> sol5:=sol4;
```

$$sol5 := \_C1\,\sin(\lambda\,x)\,(\_C3\,\mathrm{e}^{(\lambda\,y)} + \_C4\,\mathrm{e}^{(-\lambda\,y)})$$

Applying the boundary condition θ(x,0) = 0, we find that C_4 = -C_3 if the x dependence of the solution is to be preserved.

```
> eval(subs(y=0,sol5))=0;
```

$$\_C1 \sin(\lambda x)(\_C3 + \_C4) = 0$$

```
> _C4:=-_C3;
```

$$\_C4 := -\_C3$$

```
> sol6:=collect(sol5,_C3);
```

$$sol6 := \_C1 \sin(\lambda x)(e^{(\lambda y)} - e^{(-\lambda y)})\_C3$$

If we now apply the boundary condition θ(L,y) = 0, we find that sin(λL)=0, that is, λ_n = n π /L, n = 1, 2, 3, ...∞. The integer n = 0 is excluded as it results in a trivial solution.

```
> subs(x=L,sol6)=0;
```

$$\_C1 \sin(\lambda L)(e^{(\lambda y)} - e^{(-\lambda y)})\_C3 = 0$$

```
> lambda:=n*Pi/L;
```

$$\lambda := \frac{n\pi}{L}$$

```
> sol7:=convert(sol6,trig);
```

$$sol7 := 2\,\_C1 \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)\_C3$$

Combining constants C_2 and C_3 and noting that the new constant may depend on n, the general solution is expressed as the sum of an infinite number of solutions.

```
>theta(x,y):=sum(C[n]*sin(n*Pi*x/L)*sinh(n*Pi*y/L),n=1..infi
nity);
```

$$\theta(x,y) := \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$$

To determine C_n, we now apply the boundary condition, θ(x, W) = 1.

```
> 1=subs(y=W,theta(x,y));
```

$$1 = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\,\pi\,x}{L}\right) \sinh\left(\frac{n\,\pi\,W}{L}\right)$$

Because sin(nπ x/L) is orthogonal in the interval x = 0 to L, we multiply both sides of the forgoing result by sin (mπ x/L) and integrate from x = 0 to L.

```
>sum(Int(C[n]*sin(n*Pi*x/L)*sin(m*Pi*x/L)*sinh(n*Pi*y/L),
x=0..L),n=1..infinity)=Int(sin(m*Pi*x/L),x=0..L);
```

$$\sum_{n=1}^{\infty} \int_0^L C_n \sin\left(\frac{n\,\pi\,x}{L}\right) \sin\left(\frac{m\,\pi\,x}{L}\right) \sinh\left(\frac{n\,\pi\,y}{L}\right) dx = \int_0^L \sin\left(\frac{m\,\pi\,x}{L}\right) dx$$

Noting that the integral under the summation vanishes except when m = n, the constants C_n can be evaluated as follows.

```
>C[n]:=int(sin(n*Pi*x/L),x=0..L)/int(sin(n*Pi*x/L)^2*sinh(n*
Pi*W/L),x=0..L);
```

$$C_n := -\frac{2\,(\cos(n\,\pi) - 1)}{\sinh\left(\frac{n\,\pi\,W}{L}\right)(-\cos(n\,\pi)\sin(n\,\pi) + n\,\pi)}$$

The solution for θ(x,y) can now be completed by substituting for C_n.

```
> theta(x,y):=subs('C[n]'=C[n],theta(x,y));
```

$$\theta(x,y) := \sum_{n=1}^{\infty} \left(-\frac{2\,(\cos(n\,\pi) - 1)\sin\left(\frac{n\,\pi\,x}{L}\right)\sinh\left(\frac{n\,\pi\,y}{L}\right)}{\sinh\left(\frac{n\,\pi\,W}{L}\right)(-\cos(n\,\pi)\sin(n\,\pi) + n\,\pi)} \right)$$

To obtain numerical results, we put L = W = 1 and sum up the series to 200 terms. Next a three-dimensional plot of the temperature distribution in the plate is created and displayed in Figure 7.2.

```
>theta2(x,y):=subs(W=1,L=1,infinity=200,theta(x,y));
```

$$\theta 2(x, y) := \sum_{n=1}^{200} \left(- \frac{2\,(\cos(n\,\pi) - 1)\,\sin(n\,\pi\,x)\,\sinh(n\,\pi\,y)}{\sinh(n\,\pi)\,(-\cos(n\,\pi)\,\sin(n\,\pi) + n\,\pi)} \right)$$

```
> theta3:=evalf(theta2(x,y)):
> plot3d(theta3,x=0..1,y=0..1,axes=box);
```

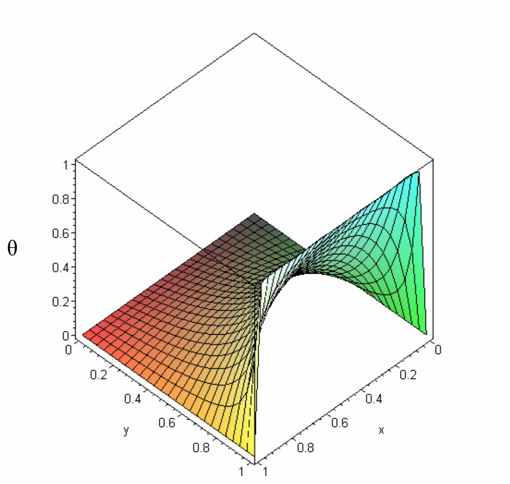


Fig 7.2: Temperature distribution

The isotherms can be plotted by using the contourplot command in the plots package. The isotherms for θ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, and 0.9 are plotted in Fig. 7.3.

```
> with(plots):
> contourplot(theta3,x=0..1,y=0..1,contours=
[seq(0.1*i,i=1..9)], axes=box,color=black);
```

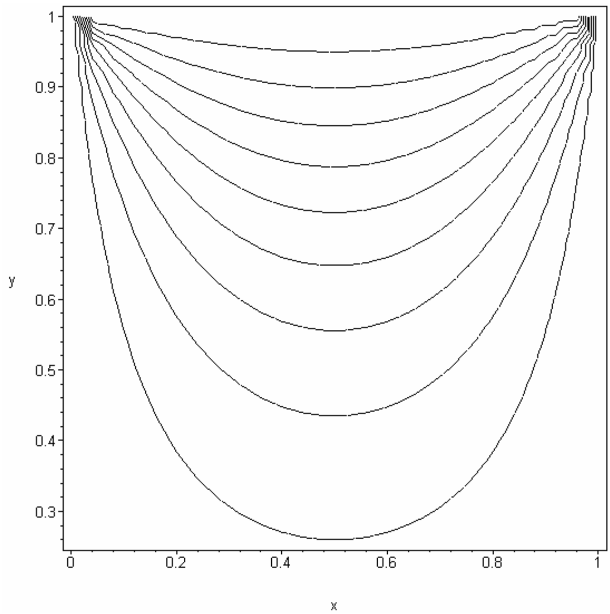


Fig 7.3: Isotherms in a rectangular plate

The temperature at any location (x,y) can be found by specifying the values of x and y in the expression for θ(x,y). Below we compute the value for x = 0.25 and y = 0.75.

```
> x:=0.25:y:=0.75:evalf(theta3,4);
```

$$0.4319$$

7.3 THE METHOD OF SEPARATION OF VARIABLES IN CYLINDRICAL COORDINATES

As an example of the method of separation of variables in cylindrical coordinates, consider a short circular cylinder of radius R and length L attached to a wall at a constant temperature T_w as shown in Figure 7.4. The lateral surface of the cylinder is exposed to a convective environment at temperature T_∞ with a uniform heat transfer coefficient, h. The top surface of the cylinder is assumed to be insulated. Heat conduction occurs in the radial direction as well as the longitudinal direction.

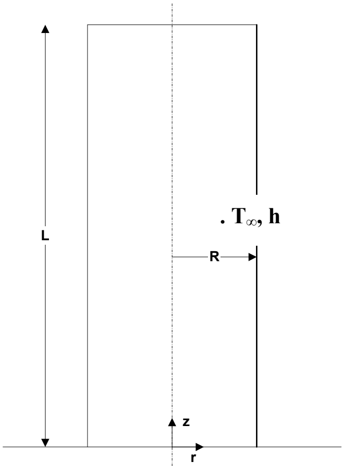


Fig 7.4. Short Circular Solid Cylinder

.

The steady-state two-dimensional heat conduction for the cylinder is

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{\partial\theta}{\partial r}+\frac{\partial^2\theta}{\partial z^2}=0 \tag{7.5}$$

where

$$\theta=\frac{T-T_\infty}{T_w-T_\infty} \tag{7.6}$$

The boundary conditions can be written as

$$\theta(z=0)=1 \tag{7.7}$$

$$\left.\frac{\partial\theta}{\partial z}\right|_{z=L}=0 \quad (insulated\ end) \tag{7.8}$$

$$\left.\frac{\partial\theta}{\partial r}\right|_{r=0}=0 \quad (thermal\ symmetry) \tag{7.9}$$

$$-k\left.\frac{\partial\theta}{\partial r}\right|_{r=R}=h\theta \tag{7.10}$$

In example 7.2, the solution to this problem is developed using the method of separation of variables.

Example 7.2: Two Dimensional Temperature Distribution in a Solid Cylinder with Surface Convection (Pin Fin)

First we create equation (7.5).

```
> restart;
```

```
> Eq1:=diff(theta(r,z),r,r)+1/r*(diff(theta(r,z),r))+
diff(theta(r,z),z,z)=0;
```

$$Eq1 := \left(\frac{\partial^2}{\partial r^2}\theta(r,z)\right)+\frac{\frac{\partial}{\partial r}\theta(r,z)}{r}+\left(\frac{\partial^2}{\partial z^2}\theta(r,z)\right)=0$$

A solution of the form

$$\theta(r,z)=G(r)H(z) \tag{7.11}$$

is assumed.

```
> theta(r,z):=G(r)*H(z);
```

$$\theta(r,z) := G(r)\,H(z)$$

where G(r) is a function of r alone and H(z) is a function of Z alone. The product form solution is substituted into the partial differential equation and divided by G(r)H(z).

```
> Eq2:=Eq1/theta(r,z);
```

$$Eq2 := \frac{\left(\frac{d^2}{dr^2}\mathrm{G}(r)\right)\mathrm{H}(z) + \frac{\left(\frac{d}{dr}\mathrm{G}(r)\right)\mathrm{H}(z)}{r} + \mathrm{G}(r)\left(\frac{d^2}{dz^2}\mathrm{H}(z)\right)}{\mathrm{G}(r)\,\mathrm{H}(z)} = 0$$

```
> Eq3:=expand(Eq2);
```

$$Eq3 := \frac{\frac{d^2}{dr^2}\mathrm{G}(r)}{\mathrm{G}(r)} + \frac{\frac{d}{dr}\mathrm{G}(r)}{\mathrm{G}(r)\,r} + \frac{\frac{d^2}{dz^2}\mathrm{H}(z)}{\mathrm{H}(z)} = 0$$

We now select the separation constant to be $-\lambda^2$. This allows us to generate two ordinary differential equations, one for G(r) and the other for H(z). These equations are then solved.

```
>Eq4:=expand(G(r)*(op(1,lhs(Eq3))+op(2,lhs(Eq3))+lambda^2));
```

$$Eq4 := \left(\frac{d^2}{dr^2}\mathrm{G}(r)\right) + \frac{\frac{d}{dr}\mathrm{G}(r)}{r} + \mathrm{G}(r)\,\lambda^2$$

```
> sol1:=dsolve(Eq4,G(r));
```

$$sol1 := \mathrm{G}(r) = \_C1\ \mathrm{BesselJ}(0, \lambda\, r) + \_C2\ \mathrm{BesselY}(0, \lambda\, r)$$

```
> Eq5:=expand(H(z)*(op(3,lhs(Eq3))-lambda^2));
```

$$Eq5 := \left(\frac{d^2}{dz^2}\mathrm{H}(z)\right) - \mathrm{H}(z)\,\lambda^2$$

```
> sol2:=dsolve(Eq5,H(z));
```

$$sol2 := \mathrm{H}(z) = \_C1\ \mathrm{e}^{(\lambda z)} + \_C2\ \mathrm{e}^{(-\lambda z)}$$

In order to avoid confusion, the constants of the z solution are renamed.

```
> sol3:=subs(_C1=_C4,_C2=_C3,sol2);
```

$$sol3 := \mathrm{H}(z) = \_C4\ \mathrm{e}^{(\lambda z)} + \_C3\ \mathrm{e}^{(-\lambda z)}$$

```
> sol4:=rhs(sol1)*rhs(sol3);
```

$$sol4 := (\_C1\ \mathrm{BesselJ}(0, \lambda\, r) + \_C2\ \mathrm{BesselY}(0, \lambda\, r))\, (\_C4\ e^{(\lambda\, z)} + \_C3\ e^{(-\lambda\, z)})$$

In order to apply the boundary condition, (7.9) we need to differentiate the fore derived solution.

```
> Eq6:=diff(sol4,r);
```

$$Eq6 := (-\_C1\ \mathrm{BesselJ}(1, \lambda\, r)\, \lambda - \_C2\ \mathrm{BesselY}(1, \lambda\, r)\, \lambda)\, (\_C4\ e^{(\lambda\, z)} + \_C3\ e^{(-\lambda\, z)})$$

Because of the fact that $Y(0) = -\infty$, the constant C_2 must equal zero to satisfy the boundary condition of zero temperature gradient at r = 0.

```
> _C2:=0;
```

$$\_C2 := 0$$

```
> sol5:=sol4;
```

$$sol5 := \_C1\ \mathrm{BesselJ}(0, \lambda\, r)\, (\_C4\ e^{(\lambda\, z)} + \_C3\ e^{(-\lambda\, z)})$$

In order to apply the boundary condition (7.8), we need to differentiate the foregoing solution with respect to z.

```
> Eq7:=diff(sol5,z);
```

$$Eq7 := \_C1\ \mathrm{BesselJ}(0, \lambda\, r)\, (\_C4\, \lambda\, e^{(\lambda\, z)} - \_C3\, \lambda\, e^{(-\lambda\, z)})$$

The application of boundary condition (7.8) gives a relationship between C_3 and C_4.

```
> eval(subs(z=L,Eq7))=0;
```

$$\_C1\ \mathrm{BesselJ}(0, \lambda\, r)\, (\_C4\, \lambda\, e^{(\lambda\, L)} - \_C3\, \lambda\, e^{(-\lambda\, L)}) = 0$$

```
> _C4:=_C3*exp(2*lambda*L);
```

$$\_C4 := \_C3\ e^{(2\, \lambda\, L)}$$

```
> sol6:=collect(sol5,_C3);
```

$$sol6 := \_C1\ \mathrm{BesselJ}(0, \lambda\, r)\, (e^{(2\, \lambda\, L)}\, e^{(\lambda\, z)} + e^{(-\lambda\, z)})\, \_C3$$

We now apply boundary condition (7.10).

```
> -k*subs(r=R,Eq6)=h*subs(r=R,sol6);
```

$$k\ \_C1\ \mathrm{BesselJ}(1, \lambda\, R)\, \lambda\, (\_C3\ e^{(2\, \lambda\, L)}\, e^{(\lambda\, z)} + \_C3\ e^{(-\lambda\, z)}) =$$
$$h\ \_C1\ \mathrm{BesselJ}(0, \lambda\, R)\, (e^{(2\, \lambda\, L)}\, e^{(\lambda\, z)} + e^{(-\lambda\, z)})\, \_C3$$

For the foregoing equality to hold true, either C_3 must be zero which results in a trivial solution or λ must take discrete values according to the following transcendental equation

$$k\lambda_n J_1(\lambda_n R) = hJ_0(\lambda_n R) \tag{7.12}$$

Combining the constants C_1 and C_2 and noting that the new constant may depend on n, the general solution is expressed as an infinite series.

$$\theta(r,z) = \sum_{n=1}^{\infty} C_n J_0(\lambda_n r)(e^{\lambda_n z} + e^{2\lambda_n L} e^{-\lambda_n z}) \tag{7.13}$$

```
>theta(r,z):=sum(C[n]*BesselJ(0,lambda[n]*r)*(exp(lambda[n]*
z)+exp(2*lambda[n]*L)*exp(-lambda[n]*z)),n=1..infinity);
```

$$\theta(r,z) := \sum_{n=1}^{\infty} C_n \, \mathrm{BesselJ}(0, \lambda_n r)\,(e^{(\lambda_n z)} + e^{(2\lambda_n L)} e^{(-\lambda_n z)})$$

To determine C_n, we may now invoke boundary condition (7.7).

```
> 1=eval(subs(z=0,theta(r,z)));
```

$$1 = \sum_{n=1}^{\infty} C_n \, \mathrm{BesselJ}(0, \lambda_n r)\,(1 + e^{(2\lambda_n L)})$$

Because $J_0(\lambda_n r)$ is orthogonal with respect to the weighting function r in the interval from 0 to R, we multiply both sides of the fore derived result by r $J_0(\lambda_m r)$ and integrate from r = 0 to R, giving

$$C_n = \frac{\int_0^R r\mathrm{J}_n(\lambda_n r)dr}{(1+e^{2\lambda_n L})\int_0^R (\mathrm{J}_n(\lambda_n r))^2 dr} \tag{7.14}$$

```
> C[n]:=int(r*(BesselJ(0,lambda[n]*r)),
r=0..R)/(1+exp(2*lambda[n]*L))/int(r*(BesselJ(0,
lambda[n]*r)^2),r=0..R);
```

$$C_n := \frac{2\,\lambda_n R\,\mathrm{BesselJ}(1, R\,\lambda_n)\sqrt{\pi}}{(1 + e^{(2\lambda_n L)})\,(\sqrt{\pi}\,R^2\,\lambda_n^{\,2}\,\mathrm{BesselJ}(0, R\,\lambda_n)^2 + \sqrt{\pi}\,R^2\,\lambda_n^{\,2}\,\mathrm{BesselJ}(1, R\,\lambda_n)^2)}$$

```
> simplify(%);
```

$$\frac{2\,\mathrm{BesselJ}(1, R\,\lambda_n)}{\lambda_n\,R\,(1+e^{(2\,\lambda_n\,L)})\,(\mathrm{BesselJ}(0, R\,\lambda_n)^2+\mathrm{BesselJ}(1, R\,\lambda_n)^2)}$$

Next C_n is substituted into the expression for θ(r,z)

```
>theta(r,z):=simplify(subs('C[n]'=C[n],theta(r,z)));
```

$$\theta(r,z) := \sum_{n=1}^{\infty}\left(\frac{2\,\mathrm{BesselJ}(1, R\,\lambda_n)\,\mathrm{BesselJ}(0, \lambda_n\,r)\,(e^{(\lambda_n\,z)}+e^{(-\lambda_n\,(-2\,L+z))})}{\lambda_n\,R\,(1+e^{(2\,\lambda_n\,L)})\,(\mathrm{BesselJ}(0, R\,\lambda_n)^2+\mathrm{BesselJ}(1, R\,\lambda_n)^2)}\right)$$

To perform the numerical computations, assume R = 1m, L = 1m, and k/h = 1m. The transcendental equation (7.12) then reduces to

$$\lambda J_1(\lambda_n) = J_0(\lambda) \tag{7.15}$$

This equation was solved in chapter 3. The first eigenvalue, λ_1 , is input. Examining the results for the next five roots shows that the appropriate interval of search for the root λ_{n+1} is λ_n+4. This knowledge is explored in computing the solution for θ(r,z)

```
> lambda[1]:=1.255783712;
```

$$\lambda_1 := 1.255783712$$

```
> for n from 2 to 10 do

> lambda[n]:=fsolve(x*BesselJ(1,x)-BesselJ(0,x),x,lambda[n-1]..lambda[n-1]+4);od:

> n:='n';
```

$$n := n$$

The following step converts the sum of an infinite series to the sum of a 10 step series.

```
> theta:=subs(infinity=10,L=1,R=1,theta(r,z));
```

$$\theta := \sum_{n=1}^{10}\left(\frac{2\,\mathrm{BesselJ}(1, \lambda_n)\,\mathrm{BesselJ}(0, \lambda_n\,r)\,(e^{(\lambda_n\,z)}+e^{(-\lambda_n\,(-2+z))})}{\lambda_n\,(1+e^{(2\,\lambda_n)})\,(\mathrm{BesselJ}(0, \lambda_n)^2+\mathrm{BesselJ}(1, \lambda_n)^2)}\right)$$

Figure 7.5 shows the dimensionless temperature distribution.

```
> plot3d(theta,r=0..1,z=0..1,axes=box);
```

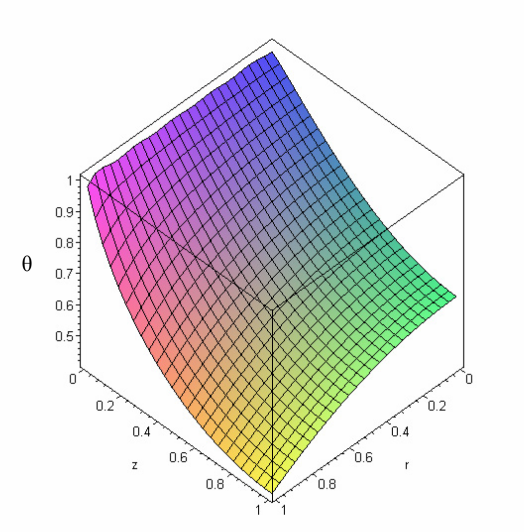


Fig 7.5: Temperature distribution in the cylinder

As an example, the radial temperature distribution is printed for z = 0.5.

```
> z:=0.5;
```

$$z := 0.5$$

```
> for r from 0 by 0.2 to 1 do print(r,theta);od;
```

0, 0.7304828181
0.2, 0.7232571368
0.4, 0.7007302175
0.6, 0.6602313316
0.8, 0.5973666694
1.0, 0.5084507262

7.4 THE METHOD OF SEPARATION OF VARIABLES IN SPHERICAL COORDINATES

To illustrate the method of separation of variables in spherical coordinates, consider a liquid droplet formed on a cooled horizontal surface as a result of condensation as shown in Fig 7.6. This situation has been described in [2]. The temperature on the outside surface of the droplet is T_s. The base temperature of the droplet is $T_w < T_s$.

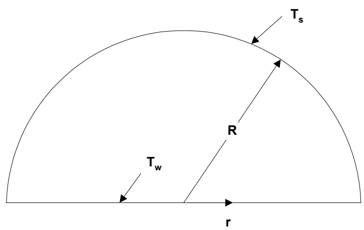


Fig 7.6. Hemispherical liquid droplet

Assuming all convection effects to be negligible and the mass of the droplet to be constant, the temperature distribution in the droplet can be described by the two dimensional steady state conduction equation in r and θ directions, that is,

$$r^2 \frac{\partial^2 \phi}{\partial r^2} + 2r \frac{\partial \phi}{\partial r} + \frac{\partial^2 \phi}{\partial \theta^2} + \cot\theta \frac{\partial \phi}{\partial \theta} = 0 \tag{7.16}$$

where $\phi = T - T_w$. The two easily recognized boundary conditions are

$$r = R, \qquad \phi = T_s - T_w = \phi_s \tag{7.17}$$

$$\theta = 0, \qquad \phi = 0 \tag{7.18}$$

Since the boundary condition on r = 0 falls on the plane θ = π/2, it is reasonable to assume equation (7.19). At θ = 0, the temperature must be bounded or thermal symmetry dictates equation (7.20). Thus the two additional boundary conditions are

$$r = 0, \qquad \phi = 0 \tag{7.19}$$

$$\theta = \pi/2, \frac{\partial \phi}{\partial \theta} = 0 \tag{7.20}$$

This problem is solved in Example 7.3.

Example 7.3 Two Dimensional Temperature Distribution in a Hemispherical Droplet

```
> restart;
```

```
> Eq1:=r^2*diff(phi(r,theta),r,r)+2*r*diff(phi(r,theta),r)+
diff(phi(r,theta),theta,theta)+cot(theta)*diff(phi(r,theta),
theta)=0;
```

$$Eq1 := r^2\left(\frac{\partial^2}{\partial r^2}\phi(r,\theta)\right)+2\,r\left(\frac{\partial}{\partial r}\phi(r,\theta)\right)+\left(\frac{\partial^2}{\partial \theta^2}\phi(r,\theta)\right)+\cot(\theta)\left(\frac{\partial}{\partial \theta}\phi(r,\theta)\right)=0$$

A product solution of the form $\phi(r,\theta)=G(r)H(\theta)$ **(7.21)** is assumed.

```
> phi(r,theta):=G(r)*H(theta);
```

$$\phi(r,\theta) := \mathrm{G}(r)\,\mathrm{H}(\theta)$$

The solution is substituted into the governing equation and the result is divided by f(r,θ). The subsequent use of expand command allows the terms containing G(r) and H(θ) to be separated.

```
> Eq2:=Eq1/phi(r,theta);
```

$$Eq2 := \frac{r^2\left(\frac{d^2}{dr^2}\mathrm{G}(r)\right)\mathrm{H}(\theta)+2\,r\left(\frac{d}{dr}\mathrm{G}(r)\right)\mathrm{H}(\theta)+\mathrm{G}(r)\left(\frac{d^2}{d\theta^2}\mathrm{H}(\theta)\right)+\cot(\theta)\,\mathrm{G}(r)\left(\frac{d}{d\theta}\mathrm{H}\right.}{\mathrm{G}(r)\,\mathrm{H}(\theta)}$$

$$=0$$

The separation constant is taken as $-\lambda^2$ = -n(n+1) where n is zero or a positive integer. The resulting ordinary differentials are then solved.

```
> Eq3:=expand(Eq2);
```

$$Eq3 := \frac{r^2\left(\frac{d^2}{dr^2}\mathrm{G}(r)\right)}{\mathrm{G}(r)}+\frac{2\,r\left(\frac{d}{dr}\mathrm{G}(r)\right)}{\mathrm{G}(r)}+\frac{\frac{d^2}{d\theta^2}\mathrm{H}(\theta)}{\mathrm{H}(\theta)}+\frac{\cot(\theta)\left(\frac{d}{d\theta}\mathrm{H}(\theta)\right)}{\mathrm{H}(\theta)}=0$$

```
> Eq4:=expand(G(r)*(op(1,lhs(Eq3))+op(2,lhs(Eq3))-(n*(n+1))));
```

$$Eq4 := r^2 \left(\frac{d^2}{dr^2} G(r)\right) + 2\, r \left(\frac{d}{dr} G(r)\right) - G(r)\, n^2 - G(r)\, n$$

```
> Eq5:=expand(H(theta)*(op(3,lhs(Eq3))+op(4,lhs(Eq3))+(n*(n+1))));
```

$$Eq5 := \left(\frac{d^2}{d\theta^2} H(\theta)\right) + \cot(\theta) \left(\frac{d}{d\theta} H(\theta)\right) + H(\theta)\, n^2 + H(\theta)\, n$$

Before solving the equation for H(θ), a new variable, η, defined as

$$\eta = \cos(\theta) \tag{7.22}$$

is introduced. The transformed equation for H(η) then becomes

$$(1-\eta^2)\frac{d^2H}{d\eta^2} - 2\eta\frac{dH}{d\eta} + n(n+1)H = 0 \tag{7.23}$$

The new equation for H is subsequently solved.

```
> Eq6:=(1-eta^2)*diff(H(eta),eta,eta)-2*eta*diff(H(eta),eta)+n*(n+1)*H(eta);
```

$$Eq6 := (1 - \eta^2) \left(\frac{d^2}{d\eta^2} H(\eta)\right) - 2\, \eta \left(\frac{d}{d\eta} H(\eta)\right) + n\, (n+1)\, H(\eta)$$

```
> Eq7:=dsolve(Eq6,H(eta));
```

$$Eq7 := H(\eta) = \_C1\ \mathrm{LegendreP}(n, \eta) + \_C2\ \mathrm{LegendreQ}(n, \eta)$$

```
> Eq8:=dsolve(Eq4);
```

$$Eq8 := G(r) = \_C1\, r^n + \_C2\, r^{(-n-1)}$$

```
> Eq9:=subs(_C1=_C4,_C2=_C3,Eq8);
```

$$Eq9 := G(r) = \_C4\, r^n + \_C3\, r^{(-n-1)}$$

The second term in the G(r) becomes unbounded at r = 0, so the constant $\_C_4$ must be set to zero to satisfy boundary condition (7.19).

```
> _C4:=0;
```

$$\_C4 := 0$$

```
> sol1:=rhs(Eq7)*rhs(Eq9);
```

$$sol1 := (\_C1\ \mathrm{LegendreP}(n,\eta) + \_C2\ \mathrm{LegendreQ}(n,\eta))\ \_C3\ r^{(-n-1)}$$

Noting that the range of η is between -1 and 1 for the full sphere and η = -1 blows up the second term, the constant _C2 must be set to zero.

```
> _C2:=0;
```

$$\_C2 := 0$$

```
> sol2:=sol1;
```

$$sol2 := \_C1\ \mathrm{LegendreP}(n,\eta)\ \_C3\ r^{(-n-1)}$$

```
> sol3:=sum(C[n]*P(n,eta)*r^n,n=1..infinity);
```

$$sol3 := \sum_{n=1}^{\infty} C_n\, \mathrm{P}(n,\eta)\, r^n$$

Next, boundary condition (7.17) is applied.

```
> phi[s]=subs(r=R,sol3);
```

$$\phi_s = \sum_{n=1}^{\infty} C_n\, \mathrm{P}(n,\eta)\, R^n$$

The Legendre polynomial $P_n(n,\eta)$ are orthogonal in the interval from η = 0 to η = 1 (θ = π/2 to 0). Therefore, both sides are multiplied by P(m,η) and integrated to give the constant C_n as

$$C_n = \frac{\phi_s \int_0^1 P(n,\eta)d\eta}{R^n \int_0^1 [P(n,\eta)]^2\, d\eta} \qquad (7.24)$$

```
>C[n]:=phi[s]*int(P(n,eta),eta=0..1)/int(P(n,eta)^2,eta=0..1
)/R^n;
```

$$C_n := \frac{\phi_s \int_0^1 \mathrm{P}(n,\eta)\, d\eta}{\int_0^1 \mathrm{P}(n,\eta)^2\, d\eta\, R^n}$$

```
> phi(r,eta):=subs('C[n]'=C[n],sol3);
```

$$\phi(r,\eta) := \sum_{n=1}^{\infty} \frac{\phi_s \int_0^1 P(n,\eta)\, d\eta\; P(n,\eta)\, r^n}{\int_0^1 P(n,\eta)^2\, d\eta\; R^n}$$

In evaluating f(r,η), the package orthopoly containing Legendre polynomials P(n,η) must be loaded first. Also, because of the boundary condition θ = π/2, f = 0, which is a boundary condition of the first kind, only the odd values of n must be chosen in calculating the sum. Below, a three term solution for f is developed.

```
> with(orthopoly):
> phi2(r,eta):=subs(infinity=5,P(n,eta)=P(n,eta),phi(r,
eta));
```

$$\phi 2(r,\eta) := \sum_{n=1}^{5} \frac{\phi_s \int_0^1 P(n,\eta)\, d\eta\; P(n,\eta)\, r^n}{\int_0^1 P(n,\eta)^2\, d\eta\; R^n}$$

```
> simplify(phi2(r,eta));
```

$$\frac{1}{128}\,\frac{\phi_s\, \eta\, r\,(192\,R^4 - 280\,r^2 R^2 \eta^2 + 168\,r^2 R^2 + 693\,r^4 \eta^4 - 770\,r^4 \eta^2 + 165\,r^4)}{R^5}$$

7.5 THE FINITE DIFFERENCE METHOD

At the present time, the three most popular numerical approaches for solving heat conduction problems are finite differences, finite elements and boundary elements. The finite element and the boundary elements methods are well suited for conduction analysis in irregular geometries and can be implemented with *Maple*. However, the focus in this section will be exclusively on finite differences. Although the finite difference approach does not offer the convenience of finite elements for irregular shapes it is nonetheless capable of handling such shapes as shown later.

7.5.1 Cartesian Coordinates

In the finite difference approach, the conduction region is covered by a grid consisting of intersecting lines. The points of intersection are called nodes. For a rectangular region, the grid lines are drawn parallel to the boundaries as shown in Fig 7.7. The spacing along the x and y directions are Δx and Δy, respectively. The choice of $\Delta x = \Delta y$ simplifies the calculations and will be made in the examples that appear later in this section. The accuracy of the finite difference solution depends on the size of the grid (Δx, Δy). In general, the smaller the grid size the greater the accuracy of the solution.

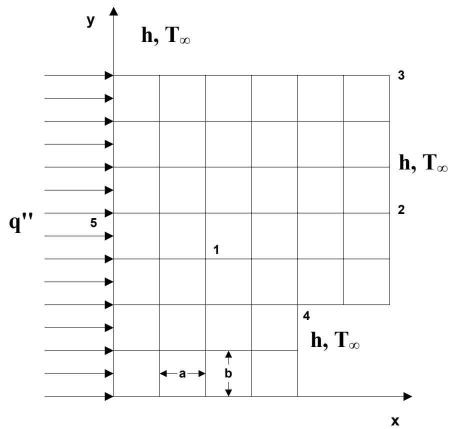


Fig 7.7. Grid size, a= Δx, b= Δy. node1: internal node, node 2: node on a plane convection surface, node 3: node at an external corner with convection, node 4: node at an internal corner with convection, node 5: node on a plane surface with uniform heat flux.

With the grid in place, the temperature distribution in the conducting medium is approximated by temperatures at discrete points (nodes). The location of a node on the grid is identified by indicies i and j, where i, j count the grid lines along the x and y directions, respectively. The origin of the coordinates will be identified either by i = 1 and j = 1 or i = 0 and j = 0 in the examples discussed in this section.

By establishing an energy balance over the control volume, a relationship between the temperature at the node under consideration and the temperature at its neighboring nodes can be derived. Such derivations are available in every heat transfer textbook. A summary of such relations is provided here for the five nodes shown in Fig. 7.7. It is assumed that there is no internal heat generation in the solid.

- Internal node

$$-T_{i,j} + \frac{1}{4}\left(T_{i+1,j} + T_{i,j+1} + T_{i-1,j} + T_{i,j-1}\right) = 0 \tag{7.25}$$

- Node on a planar convecting surface

$$2T_{i-1,j} + T_{i,j+1} + T_{i,j-1} + \frac{2h\Delta x}{k} T_{\infty} - 2\left(\frac{h\Delta x}{k} + 2\right) T_{i,j} = 0 \tag{7.26}$$

- External corner node with convection

$$T_{i-1,j} + T_{i,j-1} + \frac{2h\Delta x}{k} T_{\infty} - 2\left(\frac{h\Delta x}{k} + 1\right) T_{i,j} = 0 \tag{7.27}$$

- Internal corner node with convection

$$2\left(T_{i-1,j} + T_{i,j+1}\right) + T_{i+1,j} + T_{i,j-1} + \frac{2h\Delta x}{k} T_{\infty} - 2\left(3 + \frac{h\Delta x}{k}\right) T_{i,j} = 0 \tag{7.28}$$

- Node on a plane surface with uniform heat flux

$$2T_{i+1,j} + T_{i,j+1} + T_{i,j-1} + \frac{2q''\Delta x}{k} - 4T_{i,j} = 0 \tag{7.29}$$

For an adiabatic boundary, h=0 and for an isothermal boundary h = ∞.

It is now possible to write an appropriate finite difference equation for each node on the grid. The result is a set of linear algebraic equations. The number of equations should equal the number of nodes at which the temperature is to be determined. As pointed out in chapter 3, these equations can be solved by calling the linear algebra package which then provides access to the gauss-elimination method, the Gauss-Jordan method, and the matrix inversion method. Alternatively, one can invoke the solve command and let *Maple* use the procedure incorporated into solve to find the solution. The latter approach is used in all examples in this section.
Cylindrical Coordinates

The procedure outlined in the preceding paragraphs pertained to a conducting region that fits the Cartesian coordinates system. The same approach can, in principle, be used for a cylindrical conducting region. Consider, for example, two-dimensional steady conduction in a solid cylinder in which the temperature is a function of r and z. A sample grid is illustrated in Fig. 7.8 where Δ r= Δ z. This figure also identifies six nodes. An energy balance for each of the six nodes gives rise to the following finite difference equations.

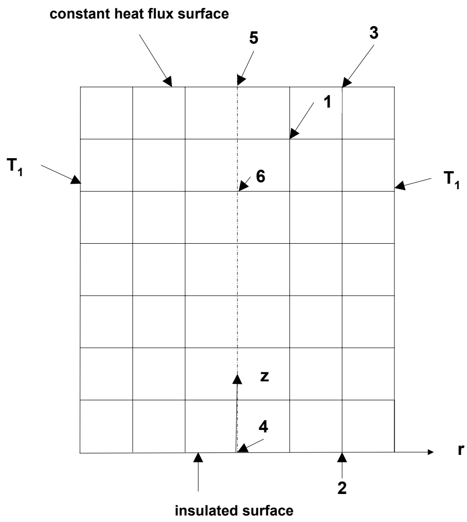


Fig 7.8. Nodes on a cylinder.

- Internal node (Node 1)

$$-T_{i,j}+\frac{1}{4}\left[\left(1-\frac{\Delta r}{2r_j}\right)T_{i,j-1}+\left(1+\frac{\Delta r}{2r_j}\right)T_{i,j+1}+T_{i+1,j}+T_{i-1,j}\right]=0 \qquad (7.30)$$

- Node on an insulated surface (bottom face, Node 2)

$$-T_{i,j}+\frac{1}{4}\left[\left(1+\frac{\Delta r}{2r_j}\right)T_{i,j+1}+\left(1-\frac{\Delta r}{2r_j}\right)T_{i,j-1}+2T_{i+1,j}\right]=0 \qquad (7.31)$$

- Node on a constant heat flux surface (Node 3)

$$-T_{i,j}+\frac{1}{4}\left[\left(1-\frac{\Delta r}{2r_j}\right)T_{i,j-1}+\left(1+\frac{\Delta r}{2r_j}\right)T_{i,j+1}+2T_{i-1,j}\right]+\frac{q^{"}\Delta r}{2k}=0 \qquad (7.32)$$

- Node at the corner of two insulated surfaces (Node 4)

$$-T_{i,j}+\frac{1}{3}\left(T_{i+1,j}+2T_{i,j+1}\right)=0,\quad i=1, j=1 \tag{7.33}$$

- Node at the corner of an insulated surface and a constant heat flux surface (Node 5)

$$-T_{i,j}+\frac{1}{3}\left(T_{i-1,j}+2T_{i,j+1}+\frac{q\Delta r}{k}\right)=0 \tag{7.34}$$

- Node on the longitudinal axis (Insulated, Node 6)

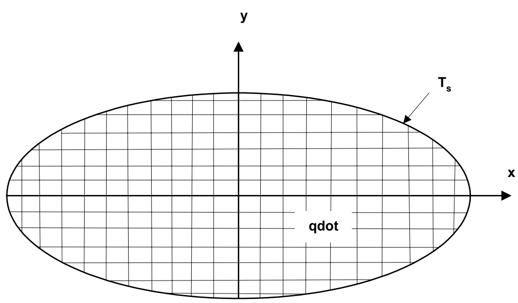
$$-T_{i,j}+\frac{1}{6}\left(T_{i+1,j}+T_{i-1,j}+4T_{i.j+1}\right)=0 \tag{7.35}$$

7.5.3 Cartesian Coordinates with Curved Boundaries

When a Cartesian grid is laid over a conducting region with a curved boundary, the nodes near the curved surface require a special treatment. For example, consider an elliptic disk with a square grid laid over it as shown in Fig 7.9. This problem has been considered by Aziz [4]. Assume that the region is generating heat at the rate of q (W/m^3) while the curved boundary is maintained at a constant temperature T_s. For two dimensional conduction in x and y directions, the governing equation is

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}+\frac{\dot{q}}{k}=0 \tag{7.36}$$

where $\theta = T - T_s$ and k is the thermal conductivity of the medium.



7.9: Elliptic Disk with a Square Grid

Thermal symmetry about the x and y axes gives the following two boundary conditions

$$\frac{\partial \theta}{\partial x}(0, y) = 0 \tag{7.37}$$

$$\frac{\partial \theta}{\partial y}(x, 0) = 0 \tag{7.38}$$

The boundary condition on the curved boundary is

$$\theta = 0 \tag{7.39}$$

A simple way of satisfying the boundary condition on the curved boundary is to approximate the curved boundary by a jagged series of steps fitting in the grid. This approach, termed as the interpolation of zero degree, is satisfactory if the grid size is sufficiently small. Let jmax[i] denote the column subscript of the rightmost grid in each row i. Since the curved boundary is described by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \tag{7.40}$$

where a and b are the major and minor axes, respectively, then jmax[i] is given by

$$j_{\max}[i] = n\left(1 - \frac{i^2}{m^2}\right)^{1/2} \tag{7.41}$$

which is truncated to the next lower integer for computational purposes.

Now that the groundwork for the finite difference method has been laid, the method is illustrated with four examples.

Example 7.4 Two-Dimensional Conduction in a Bar
A conducting bar with square cross section (0.1m x 0.1m) is shown in Fig 7.10. The upper face is kept at a uniform temperature of 100°C. Only the left half of the lower surface is cooled and maintained at 50°C. The remaining surfaces are insulated. The bar has a thermal conductivity of 25 w/m-K. The temperature distribution in the rod is to be computed using a square grid of 0.00625m x 0.00625m. It is also required to calculate the total heat transfer through the rod. It is also of interest to compare the two dimensional heat transfer with the one dimensional heat transfer if the entire upper surface is kept at 50°C. This example is adapted from Bejan [5].

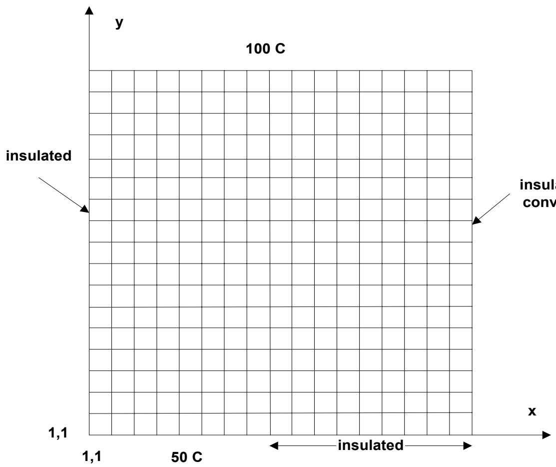


Fig 7.10:. Square Plate

The cross section of the rod is divided into 16 equal parts both along x and y giving Δx = Δy = 0.00625m. The node at the origin is labeled as 1,1. For an internal node, the residual, $d_{I,j}$ is written as

$$d_{i,j} = -T_{i,j} + \frac{1}{4}\left(T_{i+1,j} + T_{i,j+1} + T_{i-1,j} + T_{i,j-1}\right) \quad , \quad i = 2..16, j = 2..16 \tag{7.42}$$

For the upper surface, the temperature of 100°C is specified, giving

$$T_{i,17} = 100, \quad i = 1..17 \tag{7.43}$$

Similarly, for the left half of the bottom surface, we have

$$T_{i,1} = 50, \quad i = 1..9 \tag{7.44}$$

For the insulated surface at x = 0 (along y axis), the residual a_j is given by

$$a_j = -T_{1,j} + \frac{1}{4}\left(T_{1,j-1} + T_{1,j+1} + 2T_{2,j}\right), \quad j = 2..16 \tag{7.45}$$

Likewise, the residual b_j for the nodes on the insulated surface at x = 0.1 is given by

$$b_j = -T_{17,j} + \frac{1}{4}\left(2T_{16,j} + T_{17,j+1} + T_{17,j-1}\right), \quad j = 2..16 \tag{7.46}$$

The residual c $_i$ for the nodes on the insulated portion of the lower surface is

$$c_i = -T_{i,1} + \frac{1}{4}\left(2T_{i,2} + T_{i+1,1} + T_{i-1,1}\right) \tag{7.47}$$

Finally, recognizing that the node (17,1) is a node at an insulated corner, the residual, C_{17} is given by

$$C_{17} = -T_{17,1} + \frac{1}{2}\left(T_{16,1} + T_{17,2}\right) \tag{7.48}$$

Setting all residuals to zero and solving the resulting equations gives the unknown temperatures. The two dimensional heat flow through the bar can be computed by making an energy balance on the upper surface, giving

$$q_{2dtop} = k\left[\frac{1}{2}\left(T_{1,17} - T_{1,16} + T_{17,17} - T_{17,16}\right) + \sum_{m=2}^{16}\left(T_{m,17} - T_{m,16}\right)\right] \tag{7.49}$$

The one dimensional heat transfer with the entire lower surface at 50°C is given by

$$q_{1d} = k(0.1)(1)(100-50)/0.1 = 50k \tag{7.50}$$

The sequence of calculations outlined above is now coded for execution in *Maple*.

```
> restart;
> for i from 1 to 17 do
> T[i,17]:=100;od:
> for i from 1 to 9 do
> T[i,1]:=50;od:
> for j from 2 to 16 do
> a[j]:=-T[1,j]+(T[1,j-1]+T[1,j+1]+2*T[2,j])/4;
> b[j]:=-T[17,j]+(T[17,j-1]+T[17,j+1]+2*T[16,j])/4;od:
> for i from 10 to 16 do
> c[i]:=-T[i,1]+(T[i-1,1]+T[i+1,1]+2*T[i,2])/4;od:
> c[17]:=-T[17,1]+(T[16,1]+T[17,2])/2:
> for i from 2 to 16 do
> for j from 2 to 16 do
> d[i,j]:=T[i+1,j]+T[i,j+1]+T[i-1,j]+T[i,j-1]-
4*T[i,j];od:od:
> for i from 2 to 16 do
> k[i]:=d[i,1]$l=2..16;od:
>sol:=evalf(solve({a[jj]$jj=2..16,b[kk]$kk=2..16,c[mm]$mm=10
..16,c[17],k[zz]$zz=2..16})):
> assign(sol):
```

As a sample, the temperatures on the insulated portion of the lower surface are printed below.

```
> for i from 10 to 17 do
> print(T[i,1]);od;
```

58.91366928

63.65180269

66.69124296

68.80433908

70.28907990

71.28321603

71.85694598

72.04470689

```
> S1:=sum((T[m,17]-T[m,16]),m=2..16);
```

$S1 := 39.11654857$

```
> S2:=T[1,17]-T[1,16]+T[17,17]-T[17,16];
```

$S2 := 5.20282161$

```
> k:=25:
> q2dtop:=k*S1+0.5*k*S2;
```

$q2dtop := 1042.948984$

When the insulation on the lower surface of the bar is removed, heat flow becomes one dimensional. This heat flow (per unit depth) can be computed as follows.

```
> q1d:=k*0.1*(100-50)/0.1;
```

$q1d := 1250.000000$

```
> Percent_error:=100*(q1d-q2dtop)/q1d;
```

$Percent\_error := 16.56408128$

Thus the strangling of the heat flow due to the insulation reduces the heat transfer rate by 16.56 percent.

Example 7.5

Same as example 7.4 except the right face is now in contact with a fluid at a temperature of 30°C and the convective heat transfer coefficient is 50 W/m^2-K. For this case, the expression for b_j is

$$b_j = 2T_{16,j} + T_{17,j-1} + T_{17,j+1} + \frac{2h\Delta x}{k}T_a - \frac{2h\Delta x}{k}T_{17,j} \qquad (7.51)$$

```
> restart;
> h:=50:T[a]:=30:deltax:=0.00625:k:=25:
> for i from 1 to 17 do
> T[i,17]:=100;od:
> for i from 1 to 9 do
> T[i,1]:=50;od:
> for j from 2 to 16 do
> a[j]:=-T[1,j]+(T[1,j-1]+T[1,j+1]+2*T[2,j])/4;
> b[j]:=2*T[16,j]+T[17,j-1]+T[17,j+1]+(2*h*deltax*T[a]/k)-
2*((h*deltax/k)+2)*T[17,j];od:
> for i from 10 to 16 do
> c[i]:=-T[i,1]+(T[i-1,1]+T[i+1,1]+2*T[i,2])/4;od:
> c[17]:=-T[17,1]+(T[16,1]+T[17,2])/2:
> for i from 2 to 16 do
> for j from 2 to 16 do
> d[i,j]:=T[i+1,j]+T[i,j+1]+T[i-1,j]+T[i,j-1]-
4*T[i,j];od:od:
> for i from 2 to 16 do
> k[i]:=d[i,l]$l=2..16;od:
>sol:=evalf(solve({a[jj]$jj=2..16,b[kk]$kk=2..16,c[mm]$mm=10
..16,c[17],k[zz]$zz=2..16})):
> assign(sol):
```

As a sample, the temperatures on the convecting surfaces are printed.

```
> for j from 1 to 17 do
> print(T[17,j]);od;
```

67.85422727
67.80302562
68.23950938
69.05374601
70.18768808
71.59657346
73.24247795
75.09391478
77.12608393
79.32094657

81.66729572
84.16123740
86.80778065
89.62517599
92.65705329
96.01090325
100

```
> S1:=sum((T[m,17]-T[m,16]),m=2..16);
```

$S1 := 44.56326126$

```
> S2:=T[1,17]-T[1,16]+T[17,17]-T[17,16];
```

$S2 := 6.91617065$

```
> q2dtop:=25*S1+0.5*25*S2;
```

$q2dtop := 1200.533665$

With the removal of the insulation from the right face and allowing that face to transfer heat by convection to the ambient, the two-dimensional heat flow is improved from 1043W/m to 1200.5 W/m, an improvement of about 15 percent.

Example 7.6 Two-Dimensional Conduction in a Solid Cylinder
Consider the solid cylinder depicted by Fig. 7.4. The two-dimensional steady conduction in r,z direction is governed by

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} = 0 \tag{7.52}$$

Assuming the outer surface of the cylinder to be at temperature, T_1, the lower surface to be insulated and the upper surface with a uniform heat flux, the boundary conditions can be expressed as

$$r = 0, \quad \frac{\partial T}{\partial r} = 0 \quad \textit{thermal symmetry} \tag{7.53}$$

$$r = R, \quad T = T_1 \tag{7.54}$$

$$z = 0, \quad \frac{\partial T}{\partial z} = 0 \tag{7.55}$$

$$z = L, k\frac{\partial T}{\partial z} = q'' \tag{7.56}$$

For numerical calculations we take R = 1m, L = 1m, T_1 = 25°C, k = 20W/m-k, and q" = 1000W/m^2 and use the grid size Δ r = 0.25, Δ z = 0.25. Since the finite difference equations for the various types of nodes were given earlier, these need not be repeated here. Therefore, the *Maple* code and results are given directly. Note that in the *Maple* code, q" is denoted by q.

```
> restart;
> for i from 1 to 5 do
> T[i,5]:=25;od:
> r[1]:=0:deltar:=0.25:q:=1000:k:=20:
> for j from 1 to 4 do
> r[j+1]:=r[j]+deltar;od:
> for j from 2 to 4 do
> a[j]:=-T[1,j]+(2*T[2,j]+(1-deltar/2/r[j])*T[1,j-
1]+(1+deltar/2/r[j])*T[1,j+1])/4;od;
```

$$a_2 := -T_{1,2} + \frac{1}{2} T_{2,2} + 0.1250000000\ T_{1,1} + 0.3750000000\ T_{1,3}$$

$$a_3 := -T_{1,3} + \frac{1}{2} T_{2,3} + 0.1875000000\ T_{1,2} + 0.3125000000\ T_{1,4}$$

$$a_4 := -T_{1,4} + \frac{1}{2} T_{2,4} + 0.2083333334\ T_{1,3} + 7.291666670$$

```
> for j from 2 to 4 do
> b[j]:=-T[5,j]+((1-deltar/2/r[j])*T[5,j-1]+(1+
deltar/2/r[j])*T[5,j+1]+2*T[4,j])/4+q*deltar/(2*k);od;
```

$$b_2 := -T_{5,2} + 0.1250000000\ T_{5,1} + 0.3750000000\ T_{5,3} + \frac{1}{2} T_{4,2} + 6.250000000$$

$$b_3 := -T_{5,3} + 0.1875000000\ T_{5,2} + 0.3125000000\ T_{5,4} + \frac{1}{2} T_{4,3} + 6.250000000$$

$$b_4 := -T_{5,4} + 0.2083333334\ T_{5,3} + 13.54166667 + \frac{1}{2} T_{4,4}$$

```
> for i from 2 to 4 do
> c[i]:=-T[i,1]+(T[i-1,1]+T[i+1,1]+4*T[i,2])/6;od;
```

$$c_2 := -T_{2,1} + \frac{1}{6} T_{1,1} + \frac{1}{6} T_{3,1} + \frac{2}{3} T_{2,2}$$

$$c_3 := -T_{3,1} + \frac{1}{6} T_{2,1} + \frac{1}{6} T_{4,1} + \frac{2}{3} T_{3,2}$$

$$c_4 := -T_{4,1} + \frac{1}{6}T_{3,1} + \frac{1}{6}T_{5,1} + \frac{2}{3}T_{4,2}$$

```
> d[1]:=-T[1,1]+(T[2,1]+2*T[1,2])/3;
```

$$d_1 := -T_{1,1} + \frac{1}{3}T_{2,1} + \frac{2}{3}T_{1,2}$$

```
> d[2]:=-T[5,1]+(T[4,1]+2*T[5,2]+q*deltar/k)/3;
```

$$d_2 := -T_{5,1} + \frac{1}{3}T_{4,1} + \frac{2}{3}T_{5,2} + 4.166666667$$

```
> for i from 2 to 4 do
> for j from 2 to 4 do
> e[i,j]:=-T[i,j]+((1-deltar/2/r[j])*T[i,j-
1]+(1+deltar/2/r[j])*T[i,j+1]+T[i+1,j]+T[i-1,j])/4;od;od;
> for j from 2 to 4 do
> k[j]:=e[j,m]$m=2..4;od;
```

$$k_2 := -T_{2,2} + 0.1250000000\ T_{2,1} + 0.3750000000\ T_{2,3} + \frac{1}{4}T_{3,2} + \frac{1}{4}T_{1,2},$$
$$-T_{2,3} + 0.1875000000\ T_{2,2} + 0.3125000000\ T_{2,4} + \frac{1}{4}T_{3,3} + \frac{1}{4}T_{1,3},$$
$$-T_{2,4} + 0.2083333334\ T_{2,3} + 7.291666670 + \frac{1}{4}T_{3,4} + \frac{1}{4}T_{1,4}$$

$$k_3 := -T_{3,2} + 0.1250000000\ T_{3,1} + 0.3750000000\ T_{3,3} + \frac{1}{4}T_{4,2} + \frac{1}{4}T_{2,2},$$
$$-T_{3,3} + 0.1875000000\ T_{3,2} + 0.3125000000\ T_{3,4} + \frac{1}{4}T_{4,3} + \frac{1}{4}T_{2,3},$$
$$-T_{3,4} + 0.2083333334\ T_{3,3} + 7.291666670 + \frac{1}{4}T_{4,4} + \frac{1}{4}T_{2,4}$$

$$k_4 := -T_{4,2} + 0.1250000000\ T_{4,1} + 0.3750000000\ T_{4,3} + \frac{1}{4}T_{5,2} + \frac{1}{4}T_{3,2},$$
$$-T_{4,3} + 0.1875000000\ T_{4,2} + 0.3125000000\ T_{4,4} + \frac{1}{4}T_{5,3} + \frac{1}{4}T_{3,3},$$
$$-T_{4,4} + 0.2083333334\ T_{4,3} + 7.291666670 + \frac{1}{4}T_{5,4} + \frac{1}{4}T_{3,4}$$

```
> sol:=evalf(solve({a[jj]$jj=2..4,b[kk]$kk=2..4,
c[mm]$mm=2..4,d[1],d[2],k[zz]$zz=2..4}));
```

$$sol := \{T_{3,1} = 35.24236857, T_{5,2} = 50.42167844, T_{2,3} = 29.87159391, T_{3,3} = 32.32942325, T_{1,2} = 30.53917510, T_{2,4} = 27.53461775, T_{3,4} = 28.94900995, T_{1,4} = 27.12979942, T_{4,4} = 32.15356933, T_{4,1} = 41.42364651, T_{5,4} = 39.34580669, T_{2,1} = 32.02265092, T_{3,2} = 34.50197849, T_{5,1} = 51.58900113, T_{2,2} = 31.46496748, T_{1,1} = 31.03366704, T_{5,3} = 46.69130568, T_{4,3} = 37.38335277, T_{4,2} = 40.42762734, T_{1,3} = 29.13995461\}$$

Example 7.7: Two-Dimensional Conduction in an Elliptic Disk with uniform internal heat generation

As an example of two-dimensional region with a curved boundary, consider the elliptic disk shown in Fig 7.9. The mathematical description of the problem and treatment of the boundary condition on the curved surface have already been provided in section 7.5.3.

For computational purposes, the values of the various parameters are assumed as follows. $\Delta x = \Delta y = 0.5$ m, a = 10 m, b = 8m , $\dot{q} = 200$ W/m^3, k = 50 W/m.K. This gives rise to 250 simultaneous algebraic equations. To conserve space, only the temperature values at nodes along x and y axes are printed. As expected, the maximum temperature occurs at the center of the disk, This value is 76.03°C above the temperature of the isothermal curved boundary.

```
> restart;
```

```
> a:=10;b:=8;deltax:=1/2;m:=b/deltax;n:=a/deltax;qdot:=200;
k:=50;
```

$$a := 10$$
$$b := 8$$
$$deltax := \frac{1}{2}$$
$$m := 16$$
$$n := 20$$
$$qdot := 200$$
$$k := 50$$

```
> for j to n-1 do
```

```
> c[j]:=-theta[0,j]+(2*theta[1,j]+theta[0,j-
1]+theta[0,j+1]+qdot*deltax^2/k)/4;od:
> theta0y:=theta[0,tt]$tt=1..n-1:
> theta[0,n]:=0:
> for i to m-1 do
> d[i]:=-theta[i,0]+(2*theta[i,1]+theta[i-
1,0]+theta[i+1,0]+qdot*deltax^2/k)/4;od:
> theta0x:=theta[xx,0]$xx=1..m-1:
> theta[m,0]:=0:
> e[0]:=-
theta[0,0]+(theta[1,0]+theta[0,1]+qdot*deltax^2/2/k)/2:
> theta00:=theta[0,0]:
> for i from 0 to m do
> jmax[i]:=floor(n*(1-i^2/m^2)^(1/2));od:
> for i to m-1 do
> for j to jmax[i]-1 do
> f[i,j]:=-theta[i,j]+(theta[i-1,j]+theta[i+1,j]+theta[i,j-
1]+theta[i,j+1]+qdot*deltax^2/k)/4;od:od:
> for i to m-1 do
> k[i]:=f[i,h]$h=1..jmax[i]-1;od:
> for i to m-1 do
> gg[i]:=theta[i,dd]$dd=1..jmax[i]-1;od:
> thetaint:=gg[x]$x=1..m-1:
> for i to m do p:=jmax[i];
> theta[i,p]:=0;od:
> for i to m do for j to n do if  j>jmax[i] then
theta[i,j]:=0;fi;od:od:
> eqs:={c[jj]$jj=1..n-1,d[ii]$ii=1..m-1,e[0],k[zz]$zz=1..m-
1}:
> nops(eqs);
                              250
> unknowns:={theta0y,theta0x,theta00,thetaint}:
> nops(unknowns);
                              250
> s:=evalf(solve(eqs,unknowns)):
> assign(s):
```

```
> theta[0,ll]$ll=0..n;
```

76.02628799 , 75.81956306 , 75.19938465 , 74.16573997 , 72.71860084 , 70.85791334 , 68.58358498 , 65.89547461 , 62.79339431 , 59.27713475 , 55.34652562 , 51.00153489 , 46.24239133 , 41.06967178 , 35.48424221 , 29.48706463 , 23.08039388 , 16.28128380 , 9.208140368 , 2.552035092 , 0

```
> theta[pp,0]$pp=0..m;
```

76.02628799 , 75.73301292 , 74.85318411 , 73.38679270 , 71.33383031 , 68.69429701 , 65.46820670 , 61.65557557 , 57.25635786 , 52.27024658 , 46.69615830 , 40.53102710 , 33.76718368 , 26.38709080 , 18.35387118 , 9.597223336 , 0

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4. A. Aziz, Symbolic Analytic Method for Some Conduction Heat Transfer Problems, *Proceedings of the International Conference on Computational Methods, December 15-17,2004, Singapore, Springer Verlag, New York, NY, 2005.*
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Chapter 8

Transient Conduction

8.1 INTRODUCTION

The term transient or time-dependent conduction is used to describe situations in which the temperature of a heat conducting body depends on both time and spatial coordinates. Such situations arise frequently in engineering applications. For example, when a hot billet is withdrawn from a furnace and immersed in a bath of coolant, the temperature of the billet at any spatial location changes with time. Similarly, when an electronic component is activated, the accompanying heat generation triggers transient heat conduction effects.

Three models that are commonly used to study transient conduction are the lumped capacity model, the semi-infinite model and the finite size model. The lumped capacity model assumes that the temperature gradients within the body are small and, therefore, the temperature of the body is essentially a function of time. The semi-infinite model treats the body as a solid with a single identifiable surface but otherwise extending to infinity in all directions. This model is appropriate for studying the transient effects in a body of large extent such as the earth. However, it also adequately mimics the "early period" of the transient in a finite body. The finite size model provides a complete description of the transient from its onset to its termination.

This chapter is devoted to analyzing all three models using analytical as well as numerical techniques and making extensive use of *Maple* to facilitate the analysis. The analytical approaches discussed include the similarity technique, the Laplace transform method, the method of separation of variables and the method of complex temperature. On the numerical side, only the finite difference approach is discussed.

8.2 THE LUMPED CAPACITY MODEL

The lumped capacity model provides the simplest approach to solving a transient conduction problem. As already noted, the model ignores the spatial temperature variation in the body leaving time as the sole independent variable. First, the basic model is developed in which the thermal properties of the solid and the heat transfer coefficient are assumed to be constant and the heat loss from the surface occurs by pure convection. The basic model is extended to study the effects of temperature dependent specific heat, radiative heat loss, and a temperature dependent heat transfer coefficient.

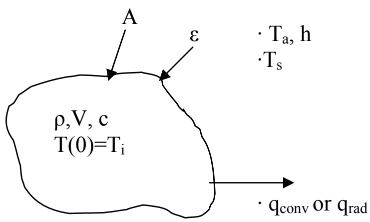

Fig. 8.1: Lumped capacity solid

8.2.1 Convective Cooling with Constant Properties

Consider a body of arbitrary shape of volume, V, surface area, A, density, ρ, and specific heat, c, initially at a uniform temperature, T_i, as shown in Fig 8.1. The heat loss from the exposed surface of the solid to the coolant is characterized by a pure convective process with a constant heat transfer coefficient, h. The fluid is assumed to have infinite thermal capacity so there is no appreciable change in the temperature of the coolant.

Because the rate of surface heat convection must equal the rate of change of internal energy of the body, the instantaneous energy balance gives

$$\rho V c \frac{dT}{dt} = -hA(T - T_a) \tag{8.1}$$

where T is the temperature of the body at any instant. The minus sign is necessary because dT/dt < 0. The initial thermal state of the body is given by

$$T(t = 0) = T_i \tag{8.2}$$

After introducing the dimensionless quantities

$$\theta = \frac{T - T_a}{T_i - T_a} \qquad and \qquad \tau = \frac{hAt}{\rho V c} \tag{8.3}$$

Eqs. (8.1) and (8.2) take the form of

$$\frac{d\theta}{dt}+\theta=0 \tag{8.4}$$

and

$$\theta(\tau=0)=1 \tag{8.5}$$

The total energy transfer, Q, over a period of time, t, expressed as a fraction of the internal energy of the solid can be determined from

$$Q=\frac{\int_0^t hA(T-T_a)dt}{\rho Vc(T_i-T_a)}=\int_0^\tau \theta d\tau \tag{8.6}$$

The quantity ρVc / hA, used to nondimensionalize time, is called the thermal time constant.

Example 8.1 Convective Cooling with Constant Properties

```
> restart;
> Eq1:=diff(theta(tau),tau)+theta(tau)=0;
```

$$Eq1 := \left(\frac{d}{d\tau}\theta(\tau)\right)+\theta(\tau)=0$$

```
> Eq2:=dsolve({Eq1,theta(0)=1},theta(tau));
```

$$Eq2 := \theta(\tau)=e^{(-\tau)}$$

```
> plot(rhs(Eq2),tau=0..5,labels=['tau','theta']);
```

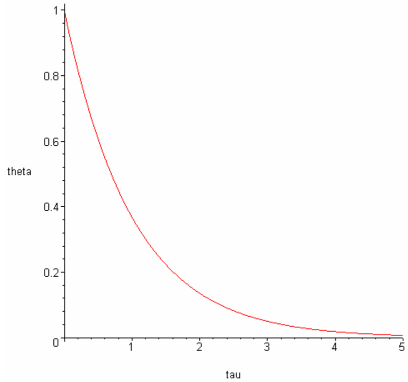

Fig. 8.2 Temperature versus time

```
> Q:=int(exp(-eta),eta=0..tau);
```

$$Q := -e^{(-\tau)} + 1$$

```
> plot(Q,tau=0..5,labels=['Q','tau']);
```

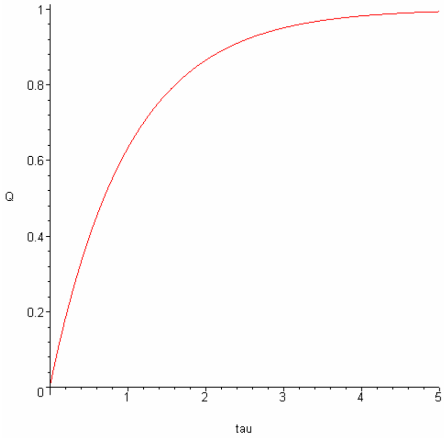

Fig 8.3 Q versus time

Figure 8.2 shows the exponential decay of the temperature, θ, with time and Fig 8.3 shows that the fraction Q increases with time and eventually attains a value of one when the body reaches thermal equilibrium with the surroundings.

8.2.2 Convective Cooling with Temperature Dependent Specific Heat

The analysis of Section 8.2.1 has been modified by Aziz and Hamad [1] to allow for linear variation of specific heat with temperature, that is,

$$c = c_a\left[T \pm \beta(T - T_a)\right] \tag{8.7}$$

where c_a is the specific heat at temperature T_a and β is the slope of the specific heat- temperature line divided by the intercept c_a. With the introduction of Eq. (8.7) into Eq. (8.1), the energy equation takes the form

$$\frac{d\theta}{d\tau}+\frac{\theta}{1\pm a\theta}=0 \tag{8.8}$$

where $a=\beta(T_1-T_a)$ and $\tau=hAt/\rho Vc_a$.

The solution to Eq. (8.8), subject to the initial conditions of Eq. (8.5), can be obtained in implicit form as

$$\ln\theta\pm a(\theta-1)=\tau \tag{8.9}$$

Example 8.2: Convective Cooling with Temperature Dependent Specific Heat.

```
>Eq3:=diff(theta(tau),tau)+theta(tau)/(1+a*theta(tau))=0;
```

$$Eq3:=\left(\frac{d}{d\tau}\theta(\tau)\right)+\frac{\theta(\tau)}{1+a\,\theta(\tau)}=0$$

```
> Eq4:=dsolve({Eq3,theta(0)=1},theta(tau));
```

$$Eq4:=\theta(\tau)=\frac{\mathrm{LambertW}(a\,\mathrm{e}^{(-\tau)}\,\mathrm{e}^{a})}{a}$$

The function, "LambertW(x)", satisfies the equation

$$ye^{y}=x \tag{8.10}$$

Because this equation has an infinite number of solutions for y for each nonzero value of x, "LambertW", has an infinite number of branches. Exactly one of these branches is analytic at 0. In *Maple*, this branch is referred to as the principle branch of "LambertW" and is denoted by "LambertW(x)." In order to plot *Eq4*, it is first converted to a function of *a* and *τ*.

```
> theta:=(a,tau)->(1/a)*LambertW(a*exp(-tau+a));;
```

$$\theta:=(a,\tau)\rightarrow\frac{\mathrm{LambertW}(a\,\mathrm{e}^{(a-\tau)})}{a}$$

```
> plot([theta(-0.6,tau),theta(-0.4,tau),theta(-
0.2,tau),limit(theta(a,tau),a=0),theta(0.2,tau),theta(0.4,ta
u),theta(0.6,tau)],tau=0..1,labels=['tau','theta'],legend=["
a=-0.6","a=-0.4","a=-0.2","a=0","a=0.2","a=0.4","a=0.6"]);
```

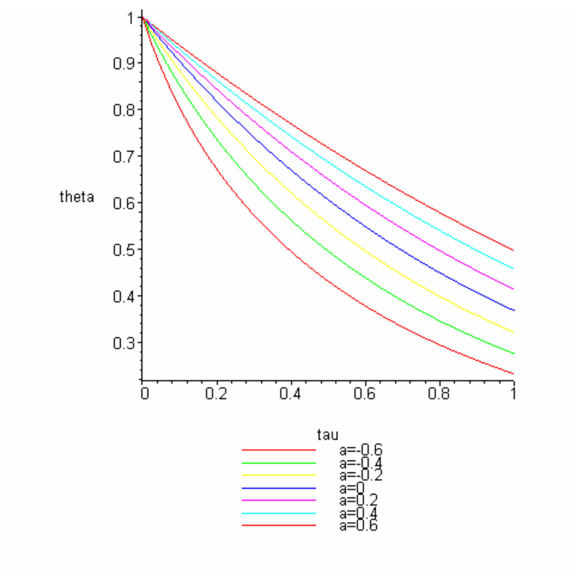

Fig. 8.4 Temperature versus time

To study the effect of varying the specific heat parameter, a, the temperature variation with time is plotted for a = -0.6, -0.4, -0.2 0.0, 0.2, 0.4, and 0.6. Note that a = 0 corresponds to the case of constant specific heat while the positive (negative) values of *a* indicate a increase (decrease) in specific heat with temperature.

Figure 8.4 shows that when the specific heat increases with temperature (a > 0), the cooling process is slower compared with the case of the specific heat ($c = c_a$). Conversely, when the specific heat decreases with temperature (a < 0), the cooling occurs at a faster rate compared with the case of constant specific heat. This behavior is consistent with our physical understanding that the greater the heat capacity of the body, the slower the cooling of the body and vice versa.

8.2.3 Radiative Cooling with Constant Properties

For pure radiative heat loss to a large surrounding sink at temperature, T_s, the energy balance is

$$\rho V c \frac{dT}{dt} = -\varepsilon \sigma A (T^4 - T_s^4) \tag{8.11}$$

where ε is the emissivity of the surface and σ is the Stefan-Boltzmann constant. Equation (8.11) can be expressed in dimensionless form as

$$\frac{d\theta}{d\tau} + \theta^4 - \theta_s^4 = 0 \tag{8.12}$$

where

$$\theta = \frac{T}{T_i}, \quad \theta_s = \frac{T_s}{T_i}, \quad and \quad \tau = \frac{\varepsilon \sigma A T_i^3 t}{\rho V c} \tag{8.13}$$

With the initial condition, θ(τ = 0) = 1, the solution of ea (8.12) is

$$\frac{1}{4\theta_s^3} \ln\left[\left(\frac{\theta_s + \theta}{\theta_s - \theta}\right)\left(\frac{\theta_s - 1}{\theta_s + 1}\right)\right] + \frac{1}{2\theta_s^3}\left[\arctan\frac{\theta}{\theta_s} - \arctan\frac{1}{\theta_s}\right] = \tau \tag{8.14}$$

Instead of asking *Maple* to solve Eq. (8.13) exactly, Example 8.3 utilized *Maple* to generate a set of numerical solutions for θ_s = 0.0, 0.2, 0.4, 0.6, and 0.8.

Example 8.3:

```
> restart;
> Eq1:=diff(theta(tau),tau)+theta(tau)^4-theta[s]^4=0;
```

$$Eq1 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^4 - \theta_s^{\ 4} = 0$$

```
> for n from 0 to 4 do
> y||n:=subs(theta[s]=0.2*n,Eq1);
```

```
> sol||n:=dsolve({y||n,theta(0)=1},theta(tau),numeric);od;
```

$$y0 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^4 = 0$$

sol0 := **proc** (*x_rkf45*) ... **end proc**

$$y1 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^4 - 0.0016 = 0$$

sol1 := **proc** (*x_rkf45*) ... **end proc**

$$y2 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^4 - 0.0256 = 0$$

sol2 := **proc** (*x_rkf45*) ... **end proc**

$$y3 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^4 - 0.1296 = 0$$

sol3 := **proc** (*x_rkf45*) ... **end proc**

$$y4 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^4 - 0.4096 = 0$$

sol4 := **proc** (*x_rkf45*) ... **end proc**

```
> for k from 0 to 4 do sol||k(1);od;
```

$$[\tau = 1., \theta(\tau) = 0.629960354313127134]$$
$$[\tau = 1., \theta(\tau) = 0.630838048265673912]$$
$$[\tau = 1., \theta(\tau) = 0.643860327844788638]$$
$$[\tau = 1., \theta(\tau) = 0.697264627073616140]$$
$$[\tau = 1., \theta(\tau) = 0.819040417626042628]$$

```
> Sol0:=tau->rhs(op(2,sol0(tau)));
```

$$Sol0 := \tau \rightarrow \mathrm{rhs}(\mathrm{op}(2, \mathrm{sol0}(\tau)))$$

```
> Sol1:=tau->rhs(op(2,sol1(tau)));
```

$$Sol1 := \tau \rightarrow \mathrm{rhs}(\mathrm{op}(2, \mathrm{sol1}(\tau)))$$

```
> Sol2:=tau->rhs(op(2,sol2(tau)));
```

$$Sol2 := \tau \rightarrow \mathrm{rhs}(\mathrm{op}(2, \mathrm{sol2}(\tau)))$$

```
> Sol3:=tau->rhs(op(2,sol3(tau)));
```

$$Sol3 := \tau \rightarrow \mathrm{rhs}(\mathrm{op}(2, \mathrm{sol3}(\tau)))$$

```
> Sol4:=tau->rhs(op(2,sol4(tau)));
```

$$Sol4 := \tau \rightarrow \mathrm{rhs}(\mathrm{op}(2, \mathrm{sol4}(\tau)))$$

```
> plot([Sol0,Sol1,Sol2,Sol3,Sol4],0..20,labels=['tau',
'theta']);
```

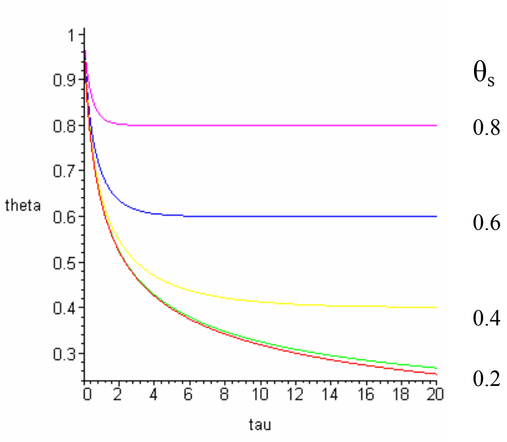

Fig 8.5: Dimensionless temperature versus tau

It can be observed from Fig. 8.5 that the fastest cooling rate occurs when the body is radiating to a sink at absolute zero ($\theta_s = 0$). As the sink temperature θ_s increases, the cooling rate gradually slows down. Also, the higher the sink temperature, the less time it takes for the body to achieve thermal equilibrium indicated by the earlier flattening of the cooling curve.

8.2.4 Convective Cooling with Temperature Dependent Heat Transfer Coefficient

If the body is cooled by natural convection, the heat transfer coefficient h is a function of the temperature difference driving the heat flow. The functional relationship for a vertical surface is

$$h = C(T - T_a)^n \tag{8.15}$$

where for the air, C = 0.59 and n = ¼ for laminar free convection and C = 0.10 and n = 1/3 for turbulent free convection. This problem was considered by Aziz and Hamad [1].

Substitution of h from Eq. (8.15) into Eq. (8.1) leads to the energy equation in dimensionless form as

$$\frac{d\theta}{dt} + \theta^{n+1} = 0 \tag{8.16}$$

where

$$\theta = \frac{T - T_a}{T_i - T_a}, \quad \tau = \frac{h_i A t}{\rho V c}, \quad and \quad h_i = C(T_i - T_a)^{\mathrm{n}} \tag{8.17}$$

With the initial condition θ(τ = 0) = 1, Eq. (8.16) integrates to

$$\theta = (1 + n\tau)^{-1/n} \tag{8.18}$$

In example 8.4, *Maple* is used to solve Eq. (8.16) subject to the initial condition, θ(τ = 0) = 1. The solution for $n = 0$ (constant heat transfer coefficient), $n = 1/3$ (turbulent natural convection), and $n = 1/4$ (laminar natural convection) are then compared graphically.

Example 8.4: Convective Cooling with Temperature Dependent Heat Transfer Coefficient

```
> restart;
> Eq1:=diff(theta(tau),tau)+theta(tau)^(1+n)=0;
```

$$Eq1 := \left(\frac{d}{d\tau}\theta(\tau)\right) + \theta(\tau)^{(1+n)} = 0$$

```
> Eq2:=dsolve({Eq1,theta(0)=1},theta(tau));
```

$$Eq2 := \theta(\tau) = \frac{1}{(n\,\tau+1)^{\left(\frac{1}{n}\right)}}$$

```
>theta:=(n,tau)>1/(tau*n+1)^(1/n);theta(0,tau):=exp(-tau);
```

$$\theta := (n,\tau) \to \frac{1}{(\tau\,n+1)^{\left(\frac{1}{n}\right)}}$$

$$\theta(0,\tau) := e^{(-\tau)}$$

```
>plot([theta(0,tau),theta(0.33,tau),theta(0.25,tau)],tau=0..
5,labels=['tau','theta'],legend=["n=0","n=1/3","n=1/4"],colo
r=black,linestyle=[SOLID,DOT,DASH]);
```

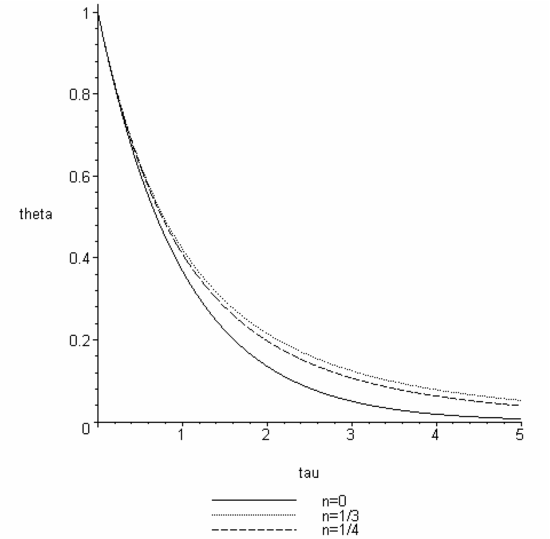

Fig. 8.6 Temperature versus time

Because the heat transfer coefficient, $h = h_i\theta^n$, h is seen to decrease as n increases. Thus the cooling occurs more slowly as n increases. This behavior is clearly shown in Fig. 8.6.

8.3 THE SEMI-INFINITE MODEL

A semi-infinite solid has a single accessible surface but extends to infinity in all other directions as shown in Fig 8.7. if the normal condition at the accessible surface is suddenly changed, the thermal disturbance propagates normal to the surface giving rise to one dimensional transient conduction. As mentioned in Section 8.1, this model only applies to solids of large extent.

However, it also describes the temperature history in a finite body in the early stages of the transient when the thermal disturbance is able to only penetrate a thin layer of the solid near the thermally disturbed surface.

The transient conduction analysis can be carried out by using either the Laplace transform method or the similarity method. The idea behind both methods is to reduce the partial differential equation governing the one-dimensional transient conduction into an ordinary differential equation so that the solution can be obtained in terms of known mathematical functions. We first illustrate the Laplace transform approach and then discuss the similarity technique.

8.3.1 The Laplace Transform Approach

The governing equation for one dimensional transient conduction is

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.19}$$

Consider the semi-infinite solid shown in Fig 8.7. Let the solid be initially at a uniform temperature throughout. At time, t = 0, the surface at x = 0 is subjected to a specified thermal disturbance (a change in temperature or a change in heat flux or exposure to a convective environment.. As the time progresses, the thermal disturbance created at the surface gradually penetrates into the solid. However, at large distances ($x \to \infty$), the solid remains immune to the disturbance and continues to remain at the initial temperature.

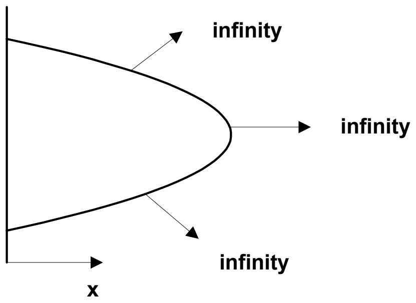

Fig 8.7 Semi-Infinite solid

8.3.1.1 Specified Surface Temperature

The temperature imposed at x = 0 for t > 0 is assumed to be of the form

$$T(0,t) = T_i + at^{n/2} \tag{8.20}$$

where *a* is a constant and *n* is a positive integer. Defining θ = T-T_i, the governing equation together with the boundary conditions can be expressed as

$$\frac{\partial^2 \theta}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.21}$$

$$\theta(x,0) = 0 \tag{8.22}$$

$$\theta(0,t) = at^{n/2} \tag{8.23}$$

$$\theta(\infty,t) = 0 \tag{8.24}$$

Example 8.5 illustrates the solution to this problem. Note that in order to satisfy Eq. (8.24) C_1 is set to zero in this example.

Example 8.5:

```
> restart;with(inttrans):
```

```
> Eq1:=diff(theta(x,t),x,x)=1/alpha*diff(theta(x,t),t);
```

$$Eq1 := \frac{\partial^2}{\partial x^2}\theta(x,t) = \frac{\frac{\partial}{\partial t}\theta(x,t)}{\alpha}$$

```
> Eq2:=laplace(Eq1,t,s);
```

$$Eq2 := \frac{\partial^2}{\partial x^2}\mathrm{laplace}(\theta(x,t),t,s) = \frac{s\,\mathrm{laplace}(\theta(x,t),t,s) - \theta(x,0)}{\alpha}$$

```
> Eq3:=subs({laplace(theta(x,t),t,s)=theta(x,s),
theta(x,0)=0},Eq2);
```

$$Eq3 := \frac{\partial^2}{\partial x^2}\theta(x,s) = \frac{s\,\theta(x,s)}{\alpha}$$

```
> Eq4:=subs(theta(x,s)=theta(x),Eq3);
```

$$Eq4 := \frac{d^2}{dx^2}\theta(x) = \frac{s\,\theta(x)}{\alpha}$$

```
> Eq5:=(dsolve(Eq4,theta(x)));
```

$$Eq5 := \theta(x) = \_C1\,\mathrm{e}^{\left(\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)} + \_C2\,\mathrm{e}^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}$$

```
> _C1:=0;
```

$$\_C1 := 0$$

```
> Eq6:=Eq5;
```

$$Eq6 := \theta(x) = \_C2\,\mathrm{e}^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}$$

```
> Eq7:=subs(_C2=F(s),Eq6);
```

$$Eq7 := \theta(x) = \mathrm{F}(s)\,\mathrm{e}^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}$$

```
> assume(n>=0):interface(showassumed=0):
F(s):=laplace(a*t^(n/2),t,s);
```

$$F(s) := a\, s^{\left(-\frac{n\sim}{2}-1\right)}\,\Gamma\left(\frac{n\sim}{2}+1\right)$$

```
> Eq8:=a*GAMMA(1+(n/2))*exp(-p*sqrt(s))/s^(1+(n/2));
```

$$Eq8 := \frac{a\,\Gamma\left(\frac{n\sim}{2}+1\right) e^{(-p\sqrt{s})}}{s^{\left(\frac{n\sim}{2}+1\right)}}$$

where $p = x/\sqrt{\alpha}$. Since *Maple's* Laplace transform library does not have the inverse of the function appearing in the foregoing equation, we add the following entry taken from Ozisik [2] to its library.

$$F(s) = \frac{e^{-p\sqrt{s}}}{s^{1+n/2}} \quad and \quad f(t) = (4t)^{n/2}\, i^n erfc\left(\frac{p}{2\sqrt{t}}\right) \qquad (8.25)$$

where i^n stands for the n^{th} repeated integral of the complementary error function.

```
> addtable(invlaplace,exp(-p*sqrt(s))/s^(1+(n/2)),
(4*t)^(n/2)*erfc(n,p/(2*sqrt(t))),s,t);
> invlaplace(exp(-p*sqrt(s))/s^(1+(n/2)),s,t);
```

$$(4\,t)^{\left(\frac{n\sim}{2}\right)} \operatorname{erfc}\left(n\sim, \frac{p}{2\sqrt{t}}\right)$$

```
> Eq9:=a*GAMMA(1+(n/2))*invlaplace(exp(-
p*sqrt(s))/s^(1+(n/2)),s,t);
```

$$Eq9 := a\,\Gamma\left(\frac{n\sim}{2}+1\right)(4\,t)^{\left(\frac{n\sim}{2}\right)} \operatorname{erfc}\left(n\sim, \frac{p}{2\sqrt{t}}\right)$$

```
> Eq10:=subs(p=x/sqrt(alpha),Eq9);
```

$$Eq10 := a\,\Gamma\left(\frac{n\sim}{2}+1\right)(4\,t)^{\left(\frac{n\sim}{2}\right)} \operatorname{erfc}\left(n\sim, \frac{x}{2\sqrt{\alpha}\sqrt{t}}\right)$$

Next, the heat flux at x = 0 is evaluated as

$$q_0'' = -k\left.\frac{\partial\theta}{\partial x}\right|_{x=0} \qquad (8.26)$$

```
> Eq11:=-k*subs(x=0,diff(Eq10,x));
```

$$Eq11 := \frac{1}{2}\frac{k\,a\,\Gamma\left(\frac{n\sim}{2}+1\right)(4\,t)^{\left(\frac{n\sim}{2}\right)}\,\mathrm{erfc}(n\sim-1,0)}{\sqrt{\alpha}\,\sqrt{t}}$$

Solutions for the three special cases, that is, n = 0, 2 and 1 can be derived from the general solution as follows.

Constant surface temperature
In this case $f(t) = T_0$, $n = 0$, and $a = T_0 - T_i$

```
> theta(x,t):=simplify(subs(n=0,Eq10));
```

$$\theta(x,t) := a\,\mathrm{erfc}\left(\frac{x}{2\,\sqrt{\alpha}\,\sqrt{t}}\right)$$

```
> q[0]:=eval(subs(n=0,Eq11));
```

$$q_0 := \frac{k\,a}{\sqrt{\pi}\,\sqrt{\alpha}\,\sqrt{t}}$$

These results show that with a step change in temperature from T_i to T_0, the temperatures within the solid monotonically approach T_0 as time increases. However, the surface heat flux decreases inversely as the square root of time.

Surface temperature varying linearly with time
In this case, $f(t) = T_i + at$ which corresponds to n = 2

```
> theta(x,t):=eval(subs(n=2,Eq10));
```

$$\theta(x,t) := 4\,a\,t\,\mathrm{erfc}\left(2, \frac{x}{2\,\sqrt{\alpha}\,\sqrt{t}}\right)$$

```
> q[2]:=radsimp(eval(subs(n=2,Eq11)));
```

$$q_2 := \frac{2\,k\,a\,\sqrt{t}}{\sqrt{\pi}\,\sqrt{\alpha}}$$

In this case, the temperature variation in the solid is a complicated function of time. The surface heat flux increases as the square root of time.

Surface Temperature varying parabolically with time
In this case, $f(t) = T_i + at^{1/2}$ which corresponds to n = 1.

```
> theta(x,t):=radsimp(eval(subs(n=1,Eq10)));
```

$$\theta(x,t) := a\sqrt{\pi}\sqrt{t}\,\mathrm{erfc}\left(1, \frac{x}{2\sqrt{\alpha}\sqrt{t}}\right)$$

```
> q[1]:=radsimp(eval(subs(n=1,Eq11)));
```

$$q_1 := \frac{k\,a\sqrt{\pi}}{2\sqrt{\alpha}}$$

Again the temperature, θ, is a complicated function of time. It is interesting to note that the surface heat flux is independent of time.

8.3.1.2 Constant Surface Heat Flux

For constant heat flux, the boundary condition at $x = 0$ takes the form

$$-k\frac{\partial \theta}{\partial x}(0,t) = q'' \tag{8.27}$$

Example 8.6

```
> restart;with(inttrans):
```

```
> addtable(invlaplace,exp(-
p*sqrt(s))/s^(1+(n/2)),(4*t)^(n/2)*erfc(n,p/(2*sqrt(t))),s,t
);
```

```
> Eq1:=diff(theta(x),x,x)=s/alpha*theta(x);
```

$$Eq1 := \frac{d^2}{dx^2}\theta(x) = \frac{s\,\theta(x)}{\alpha}$$

```
> Eq2:=dsolve(Eq1,theta(x));
```

$$Eq2 := \theta(x) = \_C1\,e^{\left(\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)} + \_C2\,e^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}$$

```
> _C1:=0;
```

$$\_C1 := 0$$

```
> Eq3:=Eq2;
```

$$Eq3 := \theta(x) = \_C2\, e^{\left(-\frac{\sqrt{s}\, x}{\sqrt{\alpha}}\right)}$$

```
> Eq4:=diff(Eq3,x);
```

$$Eq4 := \frac{d}{dx}\theta(x) = -\frac{\_C2\, \sqrt{s}\, e^{\left(-\frac{\sqrt{s}\, x}{\sqrt{\alpha}}\right)}}{\sqrt{\alpha}}$$

```
> laplace((q/k),t,s)=simplify(subs(x=0,rhs(Eq4)));
```

$$-\frac{q}{k\, s} = -\frac{\_C2\, \sqrt{s}}{\sqrt{\alpha}}$$

```
> _C2:=simplify(solve(%,_C2));
```

$$\_C2 := \frac{q\, \sqrt{\alpha}}{s^{(3/2)}\, k}$$

```
> Eq5:=Eq3;
```

$$Eq5 := \theta(x) = \frac{q\, \sqrt{\alpha}\, e^{\left(-\frac{\sqrt{s}\, x}{\sqrt{\alpha}}\right)}}{s^{(3/2)}\, k}$$

Noting that the foregoing expression corresponds to n = 1 for the Laplace transform entry added at the beginning of the example, the value of n = 1 is substituted into the inverse laplace transform.

```
> Eq6:=subs(n=1,q*sqrt(alpha)/k*invlaplace(exp(-
p*sqrt(s))/s^(1+(n/2)),s,t));
```

$$Eq6 := \frac{q\, \sqrt{\alpha}\, \sqrt{4}\, \sqrt{t}\, \mathrm{erfc}\left(1, \frac{p}{2\sqrt{t}}\right)}{k}$$

```
> theta(x,t):=radsimp(subs(p=x/sqrt(alpha),Eq6));
```

$$\theta(x, t) := \frac{2\, q\, \sqrt{\alpha}\, \sqrt{t}\, \mathrm{erfc}\left(1, \frac{x}{2\sqrt{\alpha}\, \sqrt{t}}\right)}{k}$$

This result shows that when a constant heat flux is imposed on the surface, the surface temperature increases monotonically as the square root of time.

8.3.1.3 Surface Convection

In this case, the boundary condition at x = 0 becomes

$$-k\frac{\partial T}{\partial x}(0,t) = h[T_a - T(0,t)] \tag{8.28}$$

It is more convenient to define a dimensionless temperature θ as

$$\theta = \frac{T - T_i}{T_a - T_i} \tag{8.29}$$

which transforms the boundary condition (8.28) as follows

$$-k\frac{\partial \theta}{\partial x}(0,t) = h[1-\theta(0,t)] \tag{8.30}$$

Example 8.7 provides the details of *Maple* solution.

Example 8.7:

```
> restart;
> with(inttrans):
> Eq1:=diff(theta(x),x,x)=s/alpha*theta(x);
```

$$Eq1 := \frac{d^2}{dx^2}\theta(x) = \frac{s\,\theta(x)}{\alpha}$$

```
> Eq2:=dsolve(%,theta(x));
```

$$Eq2 := \theta(x) = \_C1\, e^{\left(\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)} + \_C2\, e^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}$$

```
> _C1:=0;
```

$$\_C1 := 0$$

```
> Eq2;
```

$$\theta(x) = \_C2\, e^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}$$

```
> Eq4:=diff(Eq2,x);
```

$$Eq4 := \frac{d}{dx}\theta(x) = -\frac{\_C2\,\sqrt{s}\,e^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}}{\sqrt{\alpha}}$$

```
> -eval(subs(x=0,rhs(Eq4)))=h/k*(laplace(1,t,s)-
eval(subs(x=0,rhs(Eq2))));
```

$$\frac{\_C2\,\sqrt{s}}{\sqrt{\alpha}} = \frac{h\left(\frac{1}{s} - \_C2\right)}{k}$$

```
> _C2:=solve(%,_C2);
```

$$\_C2 := \frac{h\sqrt{\alpha}}{s^{(3/2)}\,k + h\sqrt{\alpha}\,s}$$

```
> Eq5:=Eq2;
```

$$Eq5 := \theta(x) = \frac{h\sqrt{\alpha}\,e^{\left(-\frac{\sqrt{s}\,x}{\sqrt{\alpha}}\right)}}{s^{(3/2)}\,k + h\sqrt{\alpha}\,s}$$

Denoting $h\sqrt{\alpha}/k$ by a, it is easy to express Eq5 as

```
> Eq6:=a/s/(a+sqrt(s))*exp(-p*sqrt(s));
```

(8.31)

$$Eq6 := \frac{a\,e^{(-p\sqrt{s})}}{s\,(a+\sqrt{s})}$$

Since the inverse of the foregoing expression does not appear in *Maple's* library, we add the following entry taken from Ozisik [2].

$$F(s) = \frac{ae^{-p\sqrt{s}}}{s(a+\sqrt{s})} \quad , \quad F(t) = -e^{(ap+a^2t)}erfc\left(a\sqrt{t}+\frac{b}{2\sqrt{t}}\right)+erfc\left(\frac{b}{2\sqrt{t}}\right) \qquad (8.32)$$

```
> addtable(invlaplace,exp(-p*sqrt(s))/(s*(a+sqrt(s))),-
(1/a)*exp(a*p)*exp(a^2*t)*(erfc(a*sqrt(t)+p/2/sqrt(t))+(1/a
)*erfc(p/2/sqrt(t)),s,t);
```

```
> Eq7:=a*invlaplace(exp(-p*sqrt(s))/(s*(a+sqrt(s))),s,t);
```

$$Eq7 := a\left(\frac{\mathrm{erfc}\left(\frac{p}{2\sqrt{t}}\right)}{a} - \frac{e^{(a\,p+a^2\,t)}\,\mathrm{erfc}\left(a\sqrt{t}+\frac{p}{2\sqrt{t}}\right)}{a}\right)$$

```
>theta(x,t):=simplify(subs(a=h*sqrt(alpha)/k,p=x/sqrt(alpha)
,Eq7));
```

$$\theta(x,t) := \mathrm{erfc}\left(\frac{x}{2\sqrt{\alpha}\sqrt{t}}\right) - e^{\left(\frac{h\,(x\,k+h\,\alpha\,t)}{k^2}\right)}\,\mathrm{erfc}\left(\frac{2\,h\,\alpha\,t+x\,k}{2\,k\sqrt{\alpha}\sqrt{t}}\right)$$

To study this result in detail, we recast the result into dimensionless form by defining

$$\eta = x/2\sqrt{\alpha t}\text{ , and } Bi = h\sqrt{\alpha t}/k \tag{8.33}$$

This results in the following equation:

$$\theta(\eta, Bi) = erfc(\eta) - erfc(\eta + Bi)e^{2\eta Bi^2 + Bi^2} \tag{8.34}$$

Note that Bi is a function of time based on the penetration depth $\sqrt{\alpha t}$.

In the example that follows we first create a plot of θ as a function of η for a range of values of Bi. The result is shown in Fig. 8.8. Next we plot the surface temperature as a function of Bi which appears in Fig.8.9. Finally, we consider a numerical example and study the effect of varying the thermal conductivity on the temperature and heat flux history of the solid.

Example 8.8:

```
> restart;theta(eta,Bi):=erfc(eta)-
erfc(eta+Bi)*exp(2*eta*Bi+Bi^2);
```

$$\theta(\eta, \mathrm{Bi}) := \mathrm{erfc}(\eta) - \mathrm{erfc}(\eta+\mathrm{Bi})\,e^{(2\,\eta\,\mathrm{Bi}+\mathrm{Bi}^2)}$$

```
> Eq1:=diff(theta(eta,Bi),eta);
```

$$Eq1 := -\frac{2\,e^{(-\eta^2)}}{\sqrt{\pi}} + \frac{2\,e^{(-(\eta+\mathrm{Bi})^2)}\,e^{(2\,\eta\,\mathrm{Bi}+\mathrm{Bi}^2)}}{\sqrt{\pi}} - 2\,\mathrm{erfc}(\eta+\mathrm{Bi})\,\mathrm{Bi}\,e^{(2\,\eta\,\mathrm{Bi}+\mathrm{Bi}^2)}$$

```
> theta:=unapply(theta(eta,Bi),eta,Bi);
```

$$\theta := (\eta, \mathrm{Bi}) \rightarrow \mathrm{erfc}(\eta) - \mathrm{erfc}(\eta + \mathrm{Bi})\, e^{(2\,\eta\,\mathrm{Bi} + \mathrm{Bi}^2)}$$

```
> Bi:=[0.05,0.1,0.2,0.3,0.4,0.5,1,2,3];
```

$$\mathrm{Bi} := [\,0.05, 0.1, 0.2, 0.3, 0.4, 0.5, 1, 2, 3\,]$$

```
> plot([seq(theta(eta,Bi[j]),j=1..9)],eta=0..1.5,
labels=["eta","theta(eta,Bi)"],color=black);
```

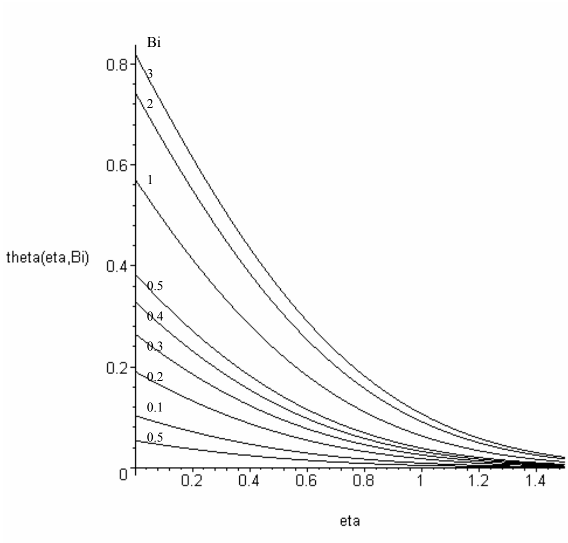

Fig 8.8: Temperature versus eta for various Bi numbers.

```
> Bi:='Bi';
```

$$Bi := Bi$$

```
> plot(theta(0,Bi),Bi=0..5,labels=["Bi","theta(0,Bi)"],
color=black);
```

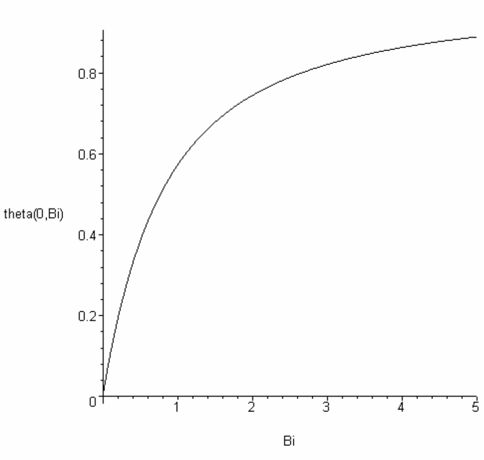

Fig 8.9: Temperature versus Biot number

```
> q[s]:=-k*(T[a]-
T[i])*simplify(subs(eta=0,Eq1)/(2*sqrt(alpha*t));
```

$$q_s := \frac{k\,(T_a - T_i)\,\mathrm{erfc}(\mathrm{Bi})\,\mathrm{Bi}\,e^{(\mathrm{Bi}^2)}}{\sqrt{\alpha\, t}}$$

As a numerical example, consider a very thick slab initially at a uniform temperature of 15°C. The thermal diffusivity of the slab is 5.6 x 10^{-6} m/s. At time t = 0,the surface at x = 0 is exposed to a hot fluid at 325°C for which the heat transfer coefficient is 100 W/m^2-K. We wish to compute the surface temperature and heat flux histories for thermal conductivities k = 2, 20, and 200W/m-K.

```
> T[i]:=15;T[a]:=325;h:=100;x:=0;alpha:=5.6*10^(-
6);eta:=x/(2*sqrt(alpha*t));;Bi:=h*sqrt(alpha*t)/k;
```

$$T_i := 15$$

$$T_a := 325$$

$$h := 100$$

$$x := 0$$

$$\alpha := 0.5600000000 \ 10^{-5}$$

$$\eta := 0.$$

$$\text{Bi} := \frac{0.2366431913 \sqrt{t}}{k}$$

```
> theta(0,Bi);
```

$$1 - \text{erfc}\left(\frac{0.2366431913 \sqrt{t}}{k}\right) e^{\left(\frac{0.05599999999\, t}{k^2}\right)}$$

```
> q[s];
```

$$31000.00000 \ \text{erfc}\left(\frac{0.2366431913 \sqrt{t}}{k}\right) e^{\left(\frac{0.05599999999\, t}{k^2}\right)}$$

```
> q[s]:=unapply(%,t,k);
```

$$q_s := (t, k) \to 31000.00000 \ \text{erfc}\left(\frac{0.2366431913 \sqrt{t}}{k}\right) e^{\left(\frac{0.05599999999\, t}{k^2}\right)}$$

```
> T:=T[i]+theta(0,Bi)*(T[a]-T[i]);
```

$$T := 325 - 310 \ \text{erfc}\left(\frac{0.2366431913 \sqrt{t}}{k}\right) e^{\left(\frac{0.05599999999\, t}{k^2}\right)}$$

```
> T:=unapply(%,t,k);
```

$$T := (t, k) \to 325 - 310 \ \text{erfc}\left(\frac{0.2366431913 \sqrt{t}}{k}\right) e^{\left(\frac{0.05599999999\, t}{k^2}\right)}$$

```
> k:=[2,20,200];
```

$$k := [2, 20, 200]$$

```
> plot([seq(T(t,k[j]),j=1..3)],t=0..300,labels=['t','T'],
color=black,linestyle=[SOLID,DASH,DOT],
legend=["k=2","k=20","k=200"]);
```

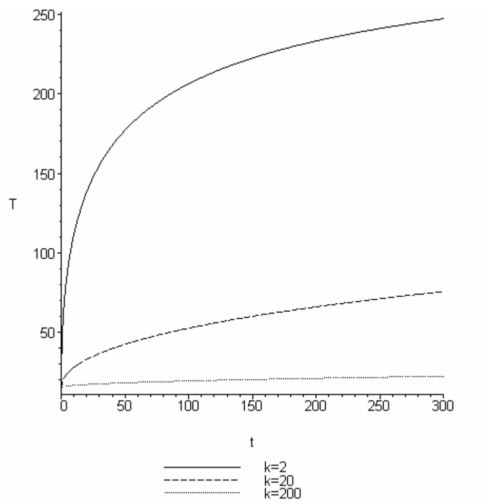

Fig 8.10: Surface temperature history

The surface temperature histories for different values of k (appearing in Fig 8.10) show that the surface responds most rapidly for k = 2 W/m-K. As k increases the response becomes progressively slower. For example, at k = 2 W/m-K, the surface temperature rises from 15°C to about 240°C in 300 seconds or five minutes after the surface is exposed to the hot fluid. When k = 200 W/m-K, the surface temperature rises to 20°C over the same period of time. This is physically understandable because for a fixed α, lower the value of k, lower the heat capacity, ρc, of the slab and hence faster the response to the hot convective environment.

```
>plot([seq(q[s](t,k[j]),j=1..3)],t=0..300,color=black,linest
yle=[SOLID,DASH,DOT],legend=["k=2","k=20","k=200"],labels=["
t","heat flux"]);
```

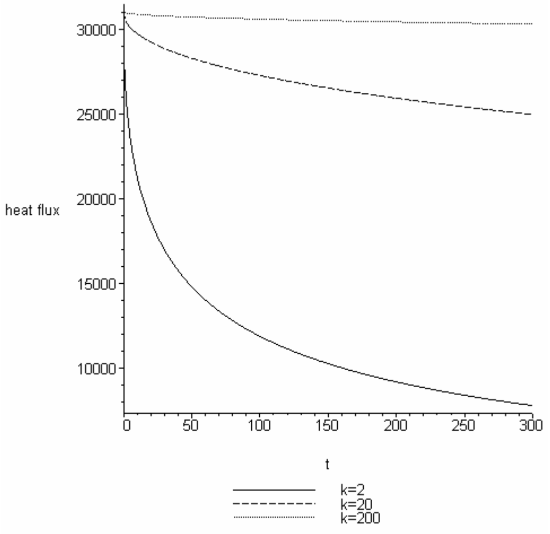

Fig 8.11: Heat flux history

The surface heat flux histories (Fig 8.11) reveals that the initially (t = 0) the heat flux equals $h(T_a-T_i)$=31000 W/m^2. Since the surface temperature rises fastest for the lowest of the three values of k, the fastest decay in heat flux occurs for the lowest value of k.

8.3.2 The Similarity Technique

The basic idea behind the similarity technique to combine the variables x and t into a single variable, η, so that the partial differential equation governing the one dimensional transient conduction can be reduced to an ordinary differential equation with η as the single independent variable. The search for the right combination for x and t or the right form of η can be systematically persued by following one of the group theoretic methods described by Hansen [3] and Seshadri and Na [4]. Another possible approach is to perform a scale analysis as described by Bejan [5] to discover the appropriate form of η.

Here a group-theoretic approach is adopted in favor of the scale analysis. Revisit the problem of one dimensional, transient conduction in a semi-infinite solid discussed in section 8.3.1. Consider the case of specified surface temperature. The equations to be solved are

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.35}$$

$$T(x,0) = T_i \quad , \quad T(\infty,t) = T_i \quad , \quad T(0,t) = T_i + at^n \tag{8.36}$$

As a first step, Equations (8.35 and 8.36) are rendered dimensionless by introducing new variables ϕ, τ, and X defined as follows

$$\phi = \frac{T - T_i}{T_i} \quad , \quad \tau = \frac{\alpha t}{L^2} \quad , \quad X = \frac{x}{L} \tag{8.37}$$

where L is a characteristic length. The resulting equations are

$$\frac{\partial^2 \phi}{\partial X^2} = \frac{\partial \phi}{\partial \tau} \tag{8.38}$$

$$\phi(X,0) = \theta_i, \phi(\infty,\tau) = 0, \phi(0,\tau) = \tau^n \tag{8.39}$$

where the constant a has been chosen for convenience to equal $T_i(\alpha/L^2)^n$.
Following Sahin [6], a one parameter, linear group transformation of the form

$$\phi = A^{\alpha_1}\bar{\phi} \quad , \quad X = A^{\alpha_2}\bar{X} \quad , \quad \tau = A^{\alpha_3}\bar{\tau} \tag{8.40}$$

is introduced into equations (8.38 and 8.39). The result is

$$A^{\alpha_1 - 2\alpha_2}\frac{\partial^2 \bar{\phi}}{\partial \bar{X}^2} = A^{\alpha_1 - \alpha_3}\frac{\partial \bar{\phi}}{\partial \bar{\tau}} \tag{8.41}$$

$$A^{\alpha_1}\bar{\phi}(0,\bar{\tau}) = A^{\alpha_3}\bar{\tau}^n \tag{8.42}$$

For equations 8.41 and 8.42 to be independent of A, the constants α_1, α_2, and α_3 must satisfy the following relations

$$\alpha_2 = \frac{1}{2}\alpha_3 \quad , \quad \alpha_1 = n\alpha_3 \tag{8.43}$$

The final step is to eliminate the parameter A from equation (8.40). This requires the determination of parameters r and s such that

$$X\tau^r = \overline{X}\,\overline{\tau}^r \quad , \quad \phi\tau^s = \overline{\phi}\,\overline{\tau}^s \tag{8.44}$$

Introducing equation 8.40 into equation 8.44 and imposing the condition of invariance (8.43), it is seen that r = -1/2 and s = -n. Thus the similarity variables η and g become

$$\eta = \frac{x}{\sqrt{\tau}} \quad , \quad g = \frac{\phi}{\tau^n} \tag{8.45}$$

With the introduction of these similarity variables, the partial differential equation (8.38) transforms into an ordinary differential equation of the form

$$\frac{d^2g}{d\eta^2} + \frac{1}{2}\eta\frac{dg}{d\eta} - ng = 0 \tag{8.46}$$

The boundary conditions (8.39) transform to

$$g(0) = 1 \quad , \quad g(\infty) = 0 \tag{8.47}$$

Note that the first two boundary conditions in equation (8.39) collapse into a single boundary condition g(∞) = 0.

In example 8.9, *Maple* is used to perform the transformation just outlined. The same example uses *Maple* to develop and study the solutions for three values of n, namely n = 0, 2, and 1.

Example 8.9:

```
> restart;
> phi:=g(eta)*tau^n;
```

$$\phi := g(\eta)\,\tau^n$$

```
> Eq1:=subs(eta=X/(sqrt(tau)),phi);
```

$$Eq1 := g\left(\frac{X}{\sqrt{\tau}}\right)\tau^n$$

$$Eq2 := \frac{\mathrm{D}(g)\left(\frac{X}{\sqrt{\tau}}\right)\tau^n}{\sqrt{\tau}}$$

```
> Eq3:=diff(Eq2,X);
```

$$Eq3 := \frac{(\mathrm{D}^{(2)})(g)\left(\frac{X}{\sqrt{\tau}}\right)\tau^n}{\tau}$$

```
> Eq4:=diff(Eq1,tau);
```

$$Eq4 := -\frac{1}{2}\frac{\mathrm{D}(g)\left(\frac{X}{\sqrt{\tau}}\right)X\,\tau^n}{\tau^{(3/2)}} + \frac{\mathrm{g}\left(\frac{X}{\sqrt{\tau}}\right)\tau^n\,n}{\tau}$$

```
> Eq3=Eq4;
```

$$\frac{(\mathrm{D}^{(2)})(g)\left(\frac{X}{\sqrt{\tau}}\right)\tau^n}{\tau} = -\frac{1}{2}\frac{\mathrm{D}(g)\left(\frac{X}{\sqrt{\tau}}\right)X\,\tau^n}{\tau^{(3/2)}} + \frac{\mathrm{g}\left(\frac{X}{\sqrt{\tau}}\right)\tau^n\,n}{\tau}$$

```
>  Eq3*tau/tau^n=op(1,Eq4)*tau/tau^n+op(2,Eq4)*tau/tau^n;
```

$$(\mathrm{D}^{(2)})(g)\left(\frac{X}{\sqrt{\tau}}\right) = -\frac{1}{2}\frac{\mathrm{D}(g)\left(\frac{X}{\sqrt{\tau}}\right)X}{\sqrt{\tau}} + \mathrm{g}\left(\frac{X}{\sqrt{\tau}}\right)n$$

```
> Eq4:=subs(X=eta*sqrt(tau),%);
```

$$Eq4 := (\mathrm{D}^{(2)})(g)(\eta) = -\frac{1}{2}\mathrm{D}(g)(\eta)\,\eta + \mathrm{g}(\eta)\,n$$

```
> Eq5:=diff(g(eta),eta,eta)+1/2*eta*diff(g(eta),eta)+n*
g(eta)=0;
```

$$Eq5 := \left(\frac{d^2}{d\eta^2}\mathrm{g}(\eta)\right) + \frac{1}{2}\eta\left(\frac{d}{d\eta}\mathrm{g}(\eta)\right) + \mathrm{g}(\eta)\,n = 0$$

Constant surface temperature, n=0

```
> Eq6:=subs(n=0,Eq5);
```

$$Eq6 := \left(\frac{d^2}{d\eta^2}\mathrm{g}(\eta)\right) + \frac{1}{2}\eta\left(\frac{d}{d\eta}\mathrm{g}(\eta)\right) = 0$$

```
> Eq7:=dsolve(Eq6,g(eta));
```

$$Eq7 := g(\eta) = \_C1 + \operatorname{erf}\left(\frac{\eta}{2}\right)\_C2$$

```
> bc1:=eval(simplify(subs(eta=0,rhs(Eq7)))-1);
```

$$bc1 := \_C1 - 1$$

```
>bc2:=eval(simplify(subs(eta=infinity,rhs(Eq7))));
```

$$bc2 := \_C1 + \_C2$$

```
> consts:=solve({bc1=0,bc2=0},{_C1,_C2});
```

$$consts := \{\_C1 = 1, \_C2 = -1\}$$

```
> assign(consts):
> Eq8:=rhs(Eq7);
```

$$Eq8 := 1 - \operatorname{erf}\left(\frac{\eta}{2}\right)$$

```
> Eq9:=subs(eta=X/sqrt(tau),Eq8);
```

$$Eq9 := 1 - \operatorname{erf}\left(\frac{X}{2\sqrt{\tau}}\right)$$

```
> sol1:=(X,tau)->1-erf(1/2*X/sqrt(tau));
```

$$sol1 := (X, \tau) \to 1 - \operatorname{erf}\left(\frac{1}{2}\frac{X}{\sqrt{\tau}}\right)$$

```
> phi:='phi';
```

$$\phi := \phi$$

```
> plot([sol1(0,tau),sol1(0.5,tau),sol1(1,tau),sol1(2,tau),
sol1(3,tau)],tau=0..5,labels=['tau','phi'],color=black,
linestyle=[SOLID,DASH,DOT,DASH,SOLID],thickness=[2,0,0,0,0],
legend=["X=0","X=0.5","X=1.0 ","X=2.0","X=3.0"]);
```

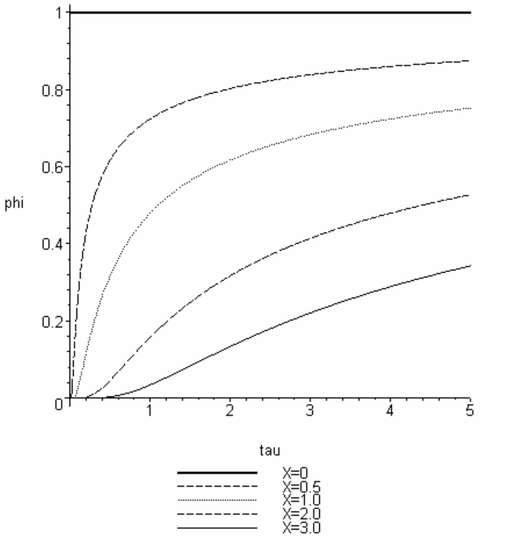

Fig 8.12: Temperature histories at different locations in the solid.

Surface temperature varying parabolically with time, n=1/2

```
> restart;
> Eq1:=diff(g(eta),eta,eta)+1/2*eta*diff(g(eta),eta)-
1/2*g(eta)=0;
```

$$Eq1 := \left(\frac{d^2}{d\eta^2}\,g(\eta)\right) + \frac{1}{2}\,\eta\left(\frac{d}{d\eta}\,g(\eta)\right) - \frac{1}{2}\,g(\eta) = 0$$

```
> Eq2:=dsolve(Eq1,g(eta));
```

$$Eq2 := g(\eta) = \_C1\,\eta + \_C2\left(e^{\left(-\frac{\eta^2}{4}\right)} + \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{\eta}{2}\right)\eta\right)$$

```
> Eq3:=subs(_C1=-1/2*sqrt(Pi),_C2=1,rhs(Eq2));
```

$$Eq3 := -\frac{\sqrt{\pi}\,\eta}{2} + e^{\left(-\frac{\eta^2}{4}\right)} + \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{\eta}{2}\right)\eta$$

```
> Eq4:=tau^(1/2)*Eq3;
```

$$Eq4 := \sqrt{\tau}\left(-\frac{\sqrt{\pi}\,\eta}{2} + e^{\left(-\frac{\eta^2}{4}\right)} + \frac{1}{2}\sqrt{\pi}\,\mathrm{erf}\left(\frac{\eta}{2}\right)\eta\right)$$

```
> Eq5:=subs(eta=X/sqrt(tau),Eq4);
```

$$Eq5 := \sqrt{\tau}\left(-\frac{\sqrt{\pi}\,X}{2\sqrt{\tau}} + e^{\left(-\frac{X^2}{4\tau}\right)} + \frac{1}{2}\frac{\sqrt{\pi}\,\mathrm{erf}\left(\frac{X}{2\sqrt{\tau}}\right)X}{\sqrt{\tau}}\right)$$

```
> sol2:=unapply(Eq5,X,tau);
```

$$sol2 := (X,\tau) \rightarrow \sqrt{\tau}\left(-\frac{1}{2}\frac{\sqrt{\pi}\,X}{\sqrt{\tau}} + e^{\left(-1/4\frac{X^2}{\tau}\right)} + \frac{1}{2}\frac{\sqrt{\pi}\,\mathrm{erf}\left(\frac{1}{2}\frac{X}{\sqrt{\tau}}\right)X}{\sqrt{\tau}}\right)$$

```
> plot([sol2(0,tau),sol2(0.5,tau),sol2(1,tau),sol2(2,tau),
sol2(3,tau)],tau=0..5,labels=['tau','phi'],color=black,
linestyle=[SOLID,DASH,DOT,DASH,SOLID],thickness=[2,0,0,0,0],
legend=["X=0","X=0.5","X=1.0 ","X=2.0","X=3.0"]);
```

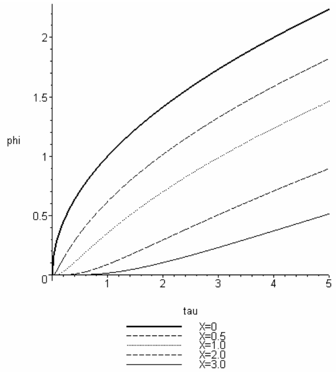

Fig 8.13: Temperature histories

Surface temperature varying linearly with time, n=1

```
> restart;
> Eq1:=diff(g(eta),eta,eta)+1/2*eta*diff(g(eta),eta)-
g(eta)=0;
```

$$Eq1 := \left(\frac{d^2}{d\eta^2} g(\eta)\right) + \frac{1}{2}\eta\left(\frac{d}{d\eta} g(\eta)\right) - g(\eta) = 0$$

```
> Eq2:=dsolve(Eq1,g(eta));
```

$$Eq2 := g(\eta) = \_C1\,(2+\eta^2) + \_C2\,(2+\eta^2)\int \frac{e^{\left(-\frac{\eta^2}{4}\right)}}{(2+\eta^2)^2}\,d\eta$$

The integral is being replaced by the function *f* in Eq2.

```
> Eq3:=_C1*(2+eta^2)+_C2*(2+eta^2)*f;
```

$$Eq3 := \_C1\,(2+\eta^2) + \_C2\,(2+\eta^2)\,f$$

```
> f:=eta->evalf(Int(exp(-p^2/4)/(p^2+2)^2,p=0..eta));
```

$$f := \eta \to \text{evalf}\left(\int_0^{\eta} \frac{e^{\left(-\frac{p^2}{4}\right)}}{(p^2+2)^2}\,dp\right)$$

```
> f(0);
```

$$0.$$

```
> f(infinity);
```

$$0.2215567314$$

```
> Eq4:=subs(_C1=1/2,_C2=-1/2/f(infinity),Eq3);
```

$$Eq4 := 1 + \frac{\eta^2}{2} - 2.256758334\,(2+\eta^2)\,f$$

```
> Eq5:=op(1,Eq4)+op(2,Eq4)+subs(f=Int(exp(-
1/4*p^2)/(p^2+2)^2,p=0..eta),op(3,Eq4));
```

$$Eq5 := 1 + \frac{\eta^2}{2} - 2.256758334\,(2+\eta^2)\int_0^{\eta} \frac{e^{\left(-\frac{p^2}{4}\right)}}{(p^2+2)^2}\,dp$$

```
> Eq6:=subs(eta=X/sqrt(tau),Eq5);
```

$$Eq6 := 1 + \frac{X^2}{2\,\tau} - 2.256758334\left(2+\frac{X^2}{\tau}\right)\int_0^{\frac{X}{\sqrt{\tau}}} \frac{e^{\left(-\frac{p^2}{4}\right)}}{(p^2+2)^2}\,dp$$

```
> sol3:=tau*unapply(Eq6,X,tau);
```

$$sol3 := \tau\left((X,\tau) \to 1 + \frac{1}{2}\frac{X^2}{\tau} - 2.256758334\left(2+\frac{X^2}{\tau}\right)\int_0^{\frac{X}{\sqrt{\tau}}} \frac{e^{\left(-\frac{p^2}{4}\right)}}{(p^2+2)^2}\,dp\right)$$

```
>plot([sol3(0,tau),sol3(0.5,tau),sol3(1,tau),sol3(2,tau),sol
3(3,tau)],tau=0..5,labels=['tau','phi'],color=black,linestyl
e=[SOLID,DASH,DOT,DASH,SOLID],thickness=[2,0,0,0,0],legend=[
"X=0","X=0.5","X=1.0 ","X=2.0","X=3.0"]);
```

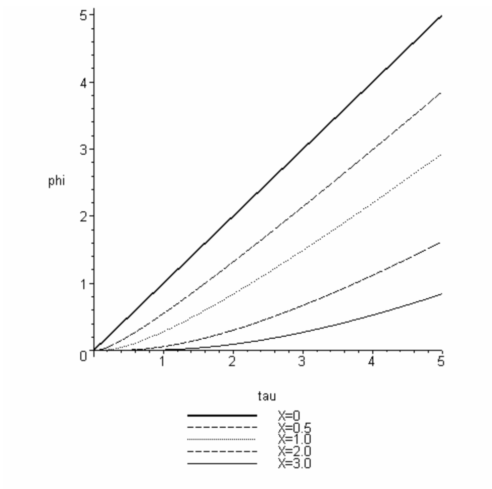

Fig 8.14: Temperature histories

Fig 8.14 depicts the temperature histories for five locations including the surface (X=0).

8.4 THE PLANE WALL WITH CONVECTION

One of the basic problems of transient conduction is the calculation of temperature history in a plane wall whose faces are suddenly exposed to a convective environment. Specifically consider a plane wall of thickness 2L made of a material with a thermal diffusivity α. The wall is initially at a uniform temperature, T_i throughout. At time $t > 0$, the two exposed surfaces of the wall are suddenly allowed to experience a convective environment at temperature T_a with heat transfer coefficient h. The temperature history of the wall is described by the following mathematical model

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \qquad (8.48)$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0 \quad , \quad -k\left.\frac{\partial T}{\partial x}\right|_{x=L} = h[T - T_\infty] \tag{8.49}$$

where x is measured from the centerline of the wall.

With the introduction of a set of dimensionless variables defined as

$$\theta = \frac{T - T_a}{T_i - T_a}, \quad X = \frac{x}{L}, \quad F_0 = \frac{\alpha t}{L^2}, \quad Bi = \frac{hL}{k} \tag{8.50}$$

the model can be recast into the following form

$$\frac{\partial^2 \theta}{\partial X^2} = \frac{\partial \theta}{\partial F_0} \tag{8.51}$$

$$\left.\frac{\partial \theta}{\partial X}\right|_{X=0} = 0 \quad , \quad \left.\frac{\partial \theta}{\partial X}\right|_{X=1} = -Bi\theta \tag{8.52}$$

The cumulative energy loss from the wall as a fraction of the initial energy content above the environment, Q, is given by

$$Q = \int_0^{Fo} \frac{\partial \theta}{\partial X}(Fo,1)dFo \tag{8.53}$$

This problem can be solved analytically by the method of separation of variables. This approach is illustrated in Example 8.10. The problem can also be solved using finite differences. Example 8.11 uses an explicit finite difference scheme to solve the problem. The implicit finite difference scheme is utilized in Example 8.12.

8.4.1 The Method of Separation of Variables

Since the method has already been discussed in detail in Chapter 7, the solution is presented in Example 8.10 without any commentary on the steps involved in the analysis.

Example 8.10:

```
> restart;
```

```
> Eq1:=diff(theta(X,Fo),X,X)-diff(theta(X,Fo),Fo)=0;
```

$$Eq1 := \left(\frac{\partial^2}{\partial X^2}\theta(X, Fo)\right) - \left(\frac{\partial}{\partial Fo}\theta(X, Fo)\right) = 0$$

```
> theta(X,Fo):=G(X)*H(Fo);
```

$$\theta(X, Fo) := G(X)\,H(Fo)$$

```
> Eq2:=Eq1/theta(X,Fo);
```

$$Eq2 := \frac{\left(\frac{d^2}{dX^2}G(X)\right)H(Fo) - G(X)\left(\frac{d}{dFo}H(Fo)\right)}{G(X)\,H(Fo)} = 0$$

```
> Eq3:=expand(Eq2);
```

$$Eq3 := \frac{\frac{d^2}{dX^2}G(X)}{G(X)} - \frac{\frac{d}{dFo}H(Fo)}{H(Fo)} = 0$$

```
> Eq4:=expand(G(X)*(op(1,lhs(Eq3))+lambda^2));
```

$$Eq4 := \left(\frac{d^2}{dX^2}G(X)\right) + G(X)\,\lambda^2$$

```
> Eq5:=dsolve(Eq4,G(X));
```

$$Eq5 := G(X) = \_C1\,\sin(\lambda\,X) + \_C2\,\cos(\lambda\,X)$$

```
> Eq6:=expand(-H(Fo)*(op(2,lhs(Eq3))-lambda^2));
```

$$Eq6 := \left(\frac{d}{dFo}H(Fo)\right) + H(Fo)\,\lambda^2$$

```
> Eq7:=dsolve(Eq6,H(Fo));
```

$$Eq7 := H(Fo) = \_C1\,e^{(-\lambda^2\,Fo)}$$

```
> Eq8:=subs(_C1=_C3,Eq7);
```

$$Eq8 := H(Fo) = \_C3\,e^{(-\lambda^2\,Fo)}$$

```
> sol:=rhs(Eq5)*rhs(Eq8);
```

$$sol := (\_C1\,\sin(\lambda\,X) + \_C2\,\cos(\lambda\,X))\,\_C3\,e^{(-\lambda^2\,Fo)}$$

```
> Eq9:=diff(sol,X);
```

$$Eq9 := (\_C1\cos(\lambda X)\lambda - \_C2\sin(\lambda X)\lambda)\,\_C3\,e^{(-\lambda^2 Fo)}$$

```
> _C1:=0;
```

$$\_C1 := 0$$

```
> Eq10:=sol;
```

$$Eq10 := \_C2\cos(\lambda X)\,\_C3\,e^{(-\lambda^2 Fo)}$$

```
> Eq11:=subs(_C2=C/_C3,Eq10);
```

$$Eq11 := C\cos(\lambda X)\,e^{(-\lambda^2 Fo)}$$

```
> subs(X=1,diff(Eq11,X))+Bi*subs(X=1,Eq11);
```

$$-C\sin(\lambda)\lambda\,e^{(-\lambda^2 Fo)} + \mathrm{Bi}\,C\cos(\lambda)\,e^{(-\lambda^2 Fo)}$$

```
> theta(X,Fo):=sum(C[n]*cos(lambda[n]*X)*exp(-
lambda[n]^2*Fo),n=1..infinity);
```

$$\theta(X, Fo) := \sum_{n=1}^{\infty} C_n \cos(\lambda_n X)\,e^{\left(-\lambda_n{}^2 Fo\right)}$$

```
> 1=simplify(subs(Fo=0, theta(X,Fo)));
```

$$1 = \sum_{n=1}^{\infty} C_n \cos(\lambda_n X)$$

```
> C[n]:=int(cos(lambda[n]*X),X=0..1)/int(cos(lambda[n]*X)^2,
X=0..1);
```

$$C_n := \frac{2\sin(\lambda_n)}{\cos(\lambda_n)\sin(\lambda_n) + \lambda_n}$$

```
> theta(X,Fo):=subs('C[n]'=C[n],theta(X,Fo));
```

$$\theta(X, Fo) := \sum_{n=1}^{\infty} \left(\frac{2\sin(\lambda_n)\cos(\lambda_n X)\,e^{\left(-\lambda_n{}^2 Fo\right)}}{\cos(\lambda_n)\sin(\lambda_n) + \lambda_n} \right)$$

```
> q:=-subs(X=1,diff(theta(X,Fo),X));
```

$$q := -\left(\sum_{n=1}^{\infty}\left(-\frac{2\sin(\lambda_n)^2\lambda_n e^{\left(-\lambda_n^2 Fo\right)}}{\cos(\lambda_n)\sin(\lambda_n)+\lambda_n}\right)\right)$$

```
> Q:=sum(2*sin(lambda[n])^2*lambda[n]*int(exp(-
lambda[n]^2*dummy),dummy=0..Fo)/(cos(lambda[n])*sin(lambda[n
])+lambda[n]),n=1..infinity);
```

$$Q := \sum_{n=1}^{\infty}\left(-\frac{2\sin(\lambda_n)^2\left(e^{\left(-\lambda_n^2 Fo\right)}-1\right)}{\lambda_n\left(\cos(\lambda_n)\sin(\lambda_n)+\lambda_n\right)}\right)$$

Now we compute numerical results for Bi=1. The discrete values of λ_n are the positive roots of the transcendental equation

$$\lambda_n \tan\lambda_n = Bi \tag{8.54}$$

The first positive root λ_1 for Bi = 1 was obtained in chapter 3. The value is λ_1 = 0.8603335890. Also note from the results in chapter 3 that the appropriate interval of search for the root of λ_{n+1} is λ_n +4.

```
> restart;
> Bi:=1;
```

$$Bi := 1$$

```
> lambda[1]:=0.8603335890;
```

$$\lambda_1 := 0.8603335890$$

```
> for n from 2 to 100 do
> lambda[n]:=fsolve(m*tan(m)-1,m,lambda[n-1]..lambda[n-
1]+4);od:
> n:='n';
```

$$n := n$$

```
>theta:=sum(2*sin(lambda[n])*cos(lambda[n]*X)*exp(-
lambda[n]^2*Fo)/(cos(lambda[n])*sin(lambda[n])+lambda[n]),n=
1..100):
```

```
> plot3d(theta,X=0..1,Fo=0..1,axes=box);
```

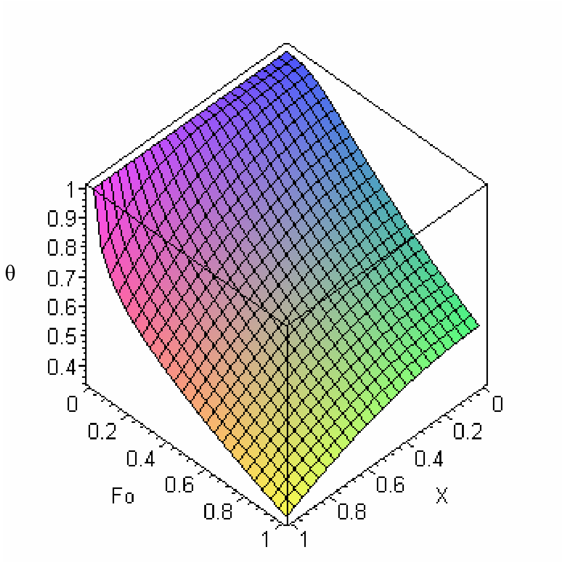

Fig 8.15: Temperature versus Position and F_0

```
> Q:=sum(2*sin(lambda[n])^2*lambda[n]*int(exp(-
lambda[n]^2*dummy),dummy=0..Fo)/(cos(lambda[n])*sin(lambda[n
])+lambda[n]),n=1..100):
```

```
> plot(Q,Fo=0..10,labels=["Fo","Q"]);
```

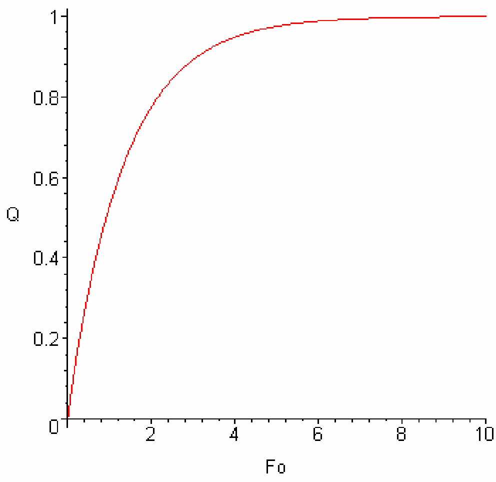

Fig 8.16: Energy loss versus F_0

A three dimensional plot of $\theta(X,F_0)$ is shown in Fig 8.15. It vividly displays the progressive cooling of the wall as the thermal disturbance penetrates towards the centerline of the wall (X=0).

The fractional cumulative energy loss from the wall as a function of dimensionless time F_0 appears in Fig 8.16. It starts with zero at $F_0 = 0$ and approaches 1 as F_0 increases. The transient has virtually died away at $F_0 = 8$ and the plane wall has reached thermal equilibrium with its surroundings.

8.4.2 Explicit Finite Difference Method

For convenience, the origin for X is located at the left face of the wall and thickness of the wall is taken as L. The boundary conditions change to

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=0} = Bi\theta \quad , \quad \left.\frac{\partial\theta}{\partial X}\right|_{X=1} = -Bi\theta \tag{8.55}$$

Also, the symbol τ is used for dimensionless time, that is $\tau = \alpha t/L^2$. The domain from X = 0 to X = 1 is divided into M equal parts giving the spacing, a, along X direction as a = 1/M. The index i identifies the location of the node on the right face of the wall as identified by i= M. The increment in time τ is denoted by b. The index j is used to track the time such that

$$\tau = j\Delta\tau \tag{8.56}$$

The explicit f9inite difference approximations for the internal and surface nodes are given by Incropera and Dewitt [7] as

Internal node: $\theta_{j,i} = F_0\theta_{j-1,i+1} + F_0\theta_{j-1,i-1} + (1-2F_0)\theta_{j-1,i}$ (8.57)

Surface node i = 0: $\theta_{j,0} = 2F_0\theta_{j-1,1} + (1-2F_0-2F_0Bia)\theta_{j-1,0}$ (8.58)

Surface node i = M: $\theta_{j,M} = 2F_0\theta_{j-1,M-1} + (1-2F_0-2F_0Bia)\theta_{j-1,M}$ (8.59)

where $F_0 = \alpha\dfrac{\Delta\tau}{(\Delta x)^2} = \dfrac{b}{a^2}$ and Bi = hL/k. The stability criterion is

$$F_0(1+Bia) \le \frac{1}{2} \quad or \quad \frac{b}{a^2}(1+Bia) \le \frac{1}{2} \tag{8.60}$$

In example 8.11, the problem for Bi = 1 and M = 20 (a = 0.05), b = 0.0005 is solved. These values satisfy the stability criterion. The total number of time intervals used is N = 2000 which corresponds to τ = 1. Because of thermal symmetry, the numerical results are displayed for the left half of the wall only. The temperature profiles are shown for four values of τ, namely τ = 0, 0.05, 0.25, and 0.5, respectively.

Example 8.11

```
> restart;
> a:=0.05;b:=0.0005;Fo:=b/a^2;Bi:=1;N:=2000;M:=20;
```

$$a := 0.05$$
$$b := 0.0005$$
$$Fo := 0.2000000000$$
$$\text{Bi} := 1$$
$$N := 2000$$
$$M := 20$$

```
> for i from 0 to M do
> theta[0,i]:=1 od:
> for j from 1 to N do
> for i from 1 to M-1 do
```

```
> theta[j,0]:=2*Fo*theta[j-1,1]+(1-2*Fo-2*Fo*Bi*a)*theta[j-
1,0];
> theta[j,M]:=2*Fo*theta[j-1,M-1]+(1-2*Fo-
2*Fo*Bi*a)*theta[j-1,M];
> theta[j,i]:=Fo*theta[j-1,i+1]+Fo*theta[j-1,i-1]+(1-
2*Fo)*theta[j-1,i];od:od:
> theta[100,s]$s=0..10;
```

0.7902103797 , 0.8275227729 , 0.8602839131 , 0.8884685102 , 0.9121683878 , 0.9315687762 , 0.9469162929 , 0.9584822737 , 0.9665255092 , 0.9712583781 , 0.9728198922

```
> plot([[seq([i,theta[0,i]],i=0..10)],[seq([i,theta[10,i]],
i=0..10)],[seq([i,theta[50,i]],i=0..10)],[seq([i,theta[100,
i]],i=0..10)]],labels=["index i","theta"]);
```

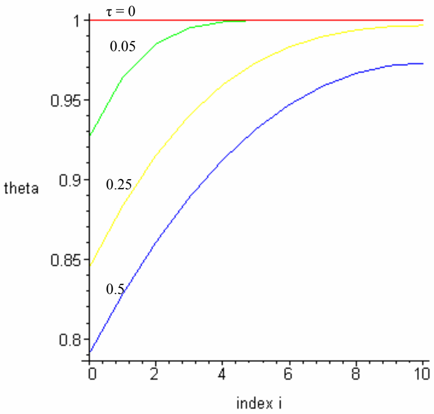

Fig 8.17: Transient temperature profiles

Fig 8.17 shows the temperature profiles in the left half of the wall at times, τ = 0, 0.05, 0.25, and 0.5.

8.4.3 Implicit Finite Difference Method

The implicit finite difference approximations for the internal and the surface nodes given by Bejan [5] are

Internal node: $\theta_{j+1,i} = \dfrac{1}{(1-2F_0)}\theta_{j,i} + \dfrac{F_0}{(1-2F_0)}\left[\theta_{j+1,i+1} + \theta_{j+1,i-1}\right]$ (8.61)

Surface node i = 0: $(1+2F_0+2F_0Bia)\theta_{j-1,0} = 2F_0\theta_{j+1,1} + \theta_{j,0}$ (8.62)

Surface node i = M: $(1+2F_0+2F_0Bia)\theta_{j-1,M} = 2F_0\theta_{j+1,M-1} + \theta_{j,M}$ (8.63)

It is evident from the implicit formulation that to determine the nodal temperatures at any instant, the corresponding nodal equations must be solved simultaneously. Thus it is necessary to solve a system of algebraic equations at each step. This is in contrast to the explicit method in which the unknown nodal temperature at new time are determined exclusively by known nodal temperatures at the previous time. However, the implicit method is unconditionally stable, that is, the solution remains stable for all sizes of space and time intervals. Computational time can therefore be considerably reduced by using a larger time interval. Although the implicit scheme is unconditionally stable, the use of a larger time interval impairs the accuracy of the results. It is therefore advisable to keep $\Delta\tau$ sufficiently small to eliminate the sensitivity of the numerical results to the size of the time interval.

In example 8.12, the implicit scheme is used to solve the problem numerically. The parameter values are a = 0.05 (M = 20), b = 0.01,
Bi = 1, and N = 100.

Example 8.12

```
> restart;
```

```
> a:=0.05;b:=0.01;Fo:=b/a^2;Bi:=1;M:=20;N:=100;
```

$$a := 0.05$$
$$b := 0.01$$
$$Fo := 4.000000000$$
$$\mathrm{Bi} := 1$$
$$M := 20$$
$$N := 100$$

```
> for i from 0 to M do
```

```
> theta[0,i]:=1:od:
> for j from 0 to N do
> Eq.[0]:=(1+2*Fo+2*Fo*a*Bi)*theta[j+1,0]-2*Fo*theta[j+1,1]-
theta[j,0]:
> Eq.[M]:=(1+2*Fo+2*Fo*a*Bi)*theta[j+1,M]-2*Fo*theta[j+1,M-
1]-theta[j,M]:
> for i from 1 to M-1 do
> Eq.[i]:=theta[j+1,i]-1/(1+2*Fo)*theta[j,i]-
Fo/(1+2*Fo)*(theta[j+1,i+1]+theta[j+1,i-1]):od:
> syst:={Eq.[mm]$mm=0..M}:
> sol:=solve(syst,{theta[j+1,mm]$mm=0..M}):
> assign(sol):od:
> theta[100,z]$z=0..M/2;
```

$$0.1563441562 , 0.1638276712 , 0.1706118559 , 0.1766677505 , 0.1819695044 , 0.1864944859 , 0.1902233793 , 0.1931402672 , 0.1952326981 , 0.1964917401 , 0.1969120188$$

These sample results match closely with those generated previously using the explicit finite difference scheme. Both the explicit and implicit results also match closely with the exact analytic solution based on the method of separation of variables (section 8.4.1).

8.5 SOLID CYLINDER WITH SUDDEN CHANGE IN SURFACE TEMPERATURE

As an example of transient conduction in a cylindrical system, consider a solid cylinder of radius r_0 initially at a temperature $T_i(r)$. At time t = 0, the temperature of the outer surface is lowered to T_0. The initial temperature $T_i(r)$ is of the form

$$T_i(r) = T_0 + aJ_0(\lambda_1 r / r_0) \tag{8.64}$$

where λ_1 is the first root of $J_0(z)=0$.

This one dimensional, transient conduction equation and the associated boundary conditions can be written as

$$\frac{\partial^2 T}{\partial r^2}+\frac{1}{r}\frac{\partial T}{\partial r}=\frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.65}$$

$$\left.\frac{\partial T}{\partial r}\right|_{(0,t)}=0 \tag{8.66}$$

$$T(r_0,t)=T_0 \tag{8.67}$$

Introducing the following dimensionless quantities

$$\theta=\frac{T-T_0}{a}\quad,\quad R=\frac{r}{r_0}\quad,\quad \tau=\frac{\alpha t}{r_0^2} \tag{8.68}$$

The mathematical statement takes the form

$$\frac{\partial^2\theta}{\partial R^2}+\frac{1}{R}\frac{\partial\theta}{\partial R}=\frac{\partial\theta}{\partial\tau} \tag{8.69}$$

$$\frac{\partial\theta}{\partial R}(0,\tau)=0,\theta(1,\tau)=0 \tag{8.70}$$

$$\theta(R,0)=J_0(\lambda_1 R) \tag{8.71}$$

The problem defined by equations 8.69 through 8.71 admits an exact analytical solution which is given by

$$\theta=J_0(\lambda_1 R)\exp(-\lambda_1^2\tau) \tag{8.72}$$

Here, however, the problem is solved using finite differences. Two schemes are used: the explicit scheme and the Crank-Nicholson scheme. For the explicit method, Ozisik [8] gives the following finite difference approximations

$$\theta_{j+1,i}=F_0(1-\frac{1}{2i})\theta_{j,i-1}+(1-2F_0)\theta_{j,i}+F_0(1+\frac{1}{2i})\theta_{j,i+1} \tag{8.73}$$

for an internal node

$$\theta_{j+1,0} = (1-4F_0)\theta_{j,0} + (4F_0)\theta_{j,1} \tag{8.74}$$

for a center node and

$$\text{i = M: } \theta_{j,M} = 0 \tag{8.75}$$

for an outer node.

For the Crank-Nicholson method, Ozisik [8] gives the following approximations where the approximation for the internal node has been corrected for typographical errors appearing in [8].

$$\begin{aligned}(1+F_0)\theta_{j+1,i} - \frac{1}{2}F_0\left(1-\frac{1}{2i}\right)\left[\theta_{j+1,i-1}+\theta_{j,i-1}\right] \\ -\frac{1}{2}F_0\left(1-\frac{1}{2i}\right)\left[\theta_{j+1,i+1}+\theta_{j,i+1}\right]-(1-F_0)\theta_{j,i}=0\end{aligned} \tag{8.76}$$

for an internal node,

$$(1+2F_0)\theta_{j+1,0} - 2F_0\theta_{j+1,1} - (1-2F_0)\theta_{j,0} = 0 \tag{8.77}$$

for the center node, i=0, and

$$\theta_{j,M} = 0 \tag{8.78}$$

for the node i=M

where $F_0 = \dfrac{\Delta\tau}{(\Delta R)^2}$

The solution based on the explicit method is developed in example 8.13 while the solution using the Crank-Nicholson method is worked out in example 8.14

Example 8.13

The solution is divided into M equal parts with center node at i= 0 and the surface node at i = M. The increment R is denoted by a, while the increment Δτ is denoted by b. The total number of steps is N. Note that the radius R at any node *i* is *a i*.

```
> restart;
> a:=0.1;b:=0.001;Fo:=b/a^2;M:=10;N:=200;
```

$$a := 0.1$$
$$b := 0.001$$
$$Fo := 0.1000000000$$
$$M := 10$$
$$N := 200$$

```
> for i from 0 to M do
> theta[0,i]:=BesselJ(0,2.404825558*a*i) od:
> for j from 0 to N do
> theta[j,M]:=0;
> theta[j+1,0]:=(1-4*Fo)*theta[j,0]+4*Fo*theta[j,1];od:
> for j from 0 to N do
> for i from 1 to M-1 do
> theta[j+1,i]:=Fo*(1-1/(2*i))*theta[j,i-1]+(1-
2*Fo)*theta[j,i]+Fo*(1+1/(2*i))*theta[j,i+1] od:od:
> theta[200,z]$z=0..10;
```

0.3155075473 , 0.3109715722 , 0.2975375494 , 0.2757887467 , 0.2466598045 , 0.2113953992 , 0.1714911850 , 0.1286227933 , 0.08456663220 , 0.04111647664 , 0

```
> Exact_value:=BesselJ(0,0)*exp(-2.404825558^2*0.2);
```

$$Exact\_value := 0.3145421489$$

The dimensionless temperature at the center at N = 200 or τ = 0.2 is 0.3155. This can be compared with the exact value of 0.3145.

Example 8.14: The Crank-Nicholson Method

```
> restart;
> a:=0.2;b:=0.005;Fo:=b/a^2;M:=5;N:=40;
```

$$a := 0.2$$
$$b := 0.005$$
$$Fo := 0.1250000000$$
$$M := 5$$
$$N := 40$$

```
> for i from 0 to M do
> theta[0,i]:=BesselJ(0,2.04048*a*i):od:
```

```
> for j from 0 to N do theta[j,M]:=0:od:
> for j from 0 to N do
> Eq.[0]:=(1+2*Fo)*theta[j+1,0]-2*Fo*theta[j+1,1]-
2*Fo*theta[j,1]-(1-2*Fo)*theta[j,0]:
> for i from 1 to M-1 do
> Eq.[i]:=(1+Fo)*theta[j+1,i]-(1/2)*Fo*(1-
1/(2*i))*(theta[j+1,i-1]+theta[j,i-1])-
(1/2)*Fo*(1+1/(2*i))*(theta[j+1,i+1]+theta[j,i+1])-(1-
Fo)*theta[j,i]:od:
> syst:={Eq.[mm]$mm=0..M-1}:
> sol:=solve(syst,{theta[j+1,mm]$mm=0..M-1}):
> assign(sol):od:
> theta[40,z]$z=0..M;
```

$$0.3665246871, 0.3459882006, 0.2873355140, 0.2002073134, 0.09891438787, 0$$

Since N = 40 corresponds to $\tau = 0.2$, the data corresponds to the results displayed in example 8.13. A comparison shows that the center temperature predicted by the Crank Nicholson is higher than the exact value (0.3665 versus 0.3145).

8.6 OSCILLATORY HEAT CONDUCTION IN A FIN

Oscillatory heat conduction is encountered in a variety of engineering applications. For example, the penetration of temperature cycles into the ground has a critical impact upon the laying of water mains, construction in cold regions and a host of geotechnical activities. Other examples of oscillatory heat conduction include heat transfer through the walls of internal combustion engines, gas turbine regenerators, and heat sinks used in cooling electronic equipment.

The temperature response in this case consists of two components, an initial transient that dies out after the execution of a sufficient number of cycles, and a steady oscillatory component which prevails indefinitely. It is the steady oscillatory component that is important in most applications. The method of complex temperature is well suited to analyze the steady oscillatory component. This method has been described by Myers [9], and more recently by Poulikakos [10]. Several applications are cited by Aziz[11].

The method can be briefly described as follows. The solution for temperature θ is written as $\theta = \theta_1(X) + \theta_2(X,\tau)$ where $\theta_2(X,\tau)$ describe the unknown steady oscillatory temperature and X and τ are the space and time variables, respectively. The first step is to introduce a new function $\theta_3(X,\tau)$ such that the equations for θ_3 are the same as those for θ_2 except that the oscillatory term, whether appearing in the equation or in the boundary condition, is shifted by π/2. The second step is to create a complex temperature whose real part is θ_2 and whose imaginary part is θ_3, that is $\Phi = \theta_2 + \theta_3$. The third step is to combine the equations for θ_2 and θ_3 to create the equation for Φ. The fourth step is to assume a solution for Φ of the form Φ = F(X)exp(i B τ) where F(X) is a function of X alone that needs to be found, and B is the dimensionless frequency of the oscillatory term appearing in the original problem. The fifth step is to substitute the assumed solution for Φ into the equation for Φ. The result of this is an ordinary differential equation for F(X), but with a complex coefficient. This equation is solved using standard techniques to obtain the solution for F and hence Φ. The final step is to extract the real part of Φ to obtain θ_2 because $\mathrm{Re}[\Phi] = \theta_2$.

To illustrate the application of the method, consider a straight fin of thickness w and length L exposed on both faces to a convective environment T_a with heat transfer coefficient h. The material of the fin has a thermal conductivity k and a thermal diffusivity α. The tip is assumed to be adiabatic while the temperature at the base is allowed to oscillate around its mean $T_{b\,m}$ with a frequency ω.

Assuming one-dimensional conduction, a unit depth, and constant thermal properties, the energy equation and the boundary conditions can be written as

$$\frac{\partial^2 T}{\partial x^2} - \frac{2h}{kw}(T - T_a) = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{8.79}$$

$$\left.\frac{\partial T}{\partial x}\right|_{x=0,t} = 0 \quad , \quad T(L,t) = T_b = T_{bm} + (T_{bm} - T_a)A\cos(\omega t) \tag{8.80}$$

where $A = (T_{b\,max} - T_{b\,m})/(T_{b\,m} - T_a)$ is the dimensionless amplitude of oscillation. Since only the steady oscillatory solution is desired, there is no need to specify the initial condition. Note that x is measured from the tip of the fin. In dimensionless form, Equations (8.79 and 8.80) become

$$\frac{\partial^2 \theta}{\partial X^2} - N^2\theta = \frac{\partial \theta}{\partial \tau} \tag{8.81}$$

$$\frac{\partial \theta}{\partial X}(0,\tau) = 0, \theta(1,\tau) = 1 + A\cos(\beta\tau) \tag{8.82}$$

where

$$\theta = \frac{T - T_a}{T_{bm} - T_a}, \quad X = \frac{x}{L}, \quad N^2 = \frac{2hL^2}{kw}, \quad \tau = \alpha\frac{t}{L^2}, \quad B = \omega\frac{L^2}{\alpha} \tag{8.83}$$

In preparation for the method of complex temperature, we write

$$\theta(X,\tau) = \theta_1(X) + \theta_2(X,\tau) \tag{8.84}$$

Substituting (8.84) into (8.81 and 8.82), the following equations are obtained for θ_1 and θ_2.

$$\frac{d^2\theta_1}{dX^2} - N^2\theta_1 = 0 \tag{8.85}$$

$$\frac{d\theta_1}{dX}(0) = 0, \theta_1(1) = 1 \tag{8.86}$$

$$\frac{\partial^2\theta_2}{\partial X^2} - N^2\theta_2 = \frac{\partial\theta_2}{\partial\tau} \tag{8.87}$$

$$\frac{\partial\theta_2}{\partial X}(0,\tau) = 0, \theta_2(1,\tau) = A\cos(B\tau) \tag{8.88}$$

Following the method of complex temperature, a new variable θ_3 is introduced such that it satisfies the following equations.

$$\frac{\partial^2\theta_3}{\partial X^2} - N^2\theta_3 = \frac{\partial\theta_3}{\partial\tau} \tag{8.89}$$

$$\frac{\partial\theta_3}{\partial X}(0,\tau) = 0, \theta_3(1,\tau) = A\sin(B\tau) \tag{8.90}$$

Next, the function Φ is defined as

$$\Phi = \theta_2 + i\theta_3 \tag{8.91}$$

Multiplying equations (8.89) and (8.90) by *i* and adding them to equations (8.87) and (8.88), the equations for Φ become

$$\frac{\partial^2\Phi}{\partial X^2} - N^2\Phi = \frac{\partial\Phi}{\partial\tau} \tag{8.92}$$

$$\frac{\partial\Phi}{\partial X}(0,\tau) = 0, \Phi(1,\tau) = A\exp(iB\tau) \tag{8.93}$$

At this stage, Φ is assumed to be of the form

$$\Phi = AF(X)e^{iB\tau} \tag{8.94}$$

where F(X) is a function of X that needs to be determined. If equation (8.94) is substituted into equations (8.92) and (8.93), the equations for F(X) are obtained as

$$\frac{d^2F}{dX^2} - (N^2 + iB) = 0 \tag{8.95}$$

$$\frac{dF}{dX}(0) = 0, F(1) = 1 \tag{8.96}$$

Once the solution for F(X) is obtained by solving equations (8.95) and (8.96), it can be substituted in equation (8.94) to give Φ. Extracting the real part of Φ gives θ_2. The extraction of the real part is easily done by invoking the **evalc(**Re(phi)**)** command.

In example 8.15, we use *Maple* to solve for θ_1 and θ_2. Once the temperature distribution is determined, we find the heat transfer rate at the base. Sample curves for the temperature and heat transfer variations are displayed both in numerical as well as graphical forms.

Example 8.15

```
> restart;
> Eq1:=diff(theta[1](X),X,X)-N^2*theta[1](X)=0;
```

$$Eq1 := \left(\frac{d^2}{dX^2}\theta_1(X)\right) - N^2\,\theta_1(X) = 0$$

```
> Eq2:=dsolve(Eq1,theta[1](X));
```

$$Eq2 := \theta_1(X) = \_C1\,e^{(-N\,X)} + \_C2\,e^{(N\,X)}$$

```
> Eq3:=diff(Eq2,X);
```

$$Eq3 := \frac{d}{dX}\theta_1(X) = -\_C1\, N\, e^{(-N\,X)} + \_C2\, N\, e^{(N\,X)}$$

```
> bc1:=simplify(subs(X=0,rhs(Eq3)));
```

$$bc1 := -N\,\_C1 + N\,\_C2$$

```
> bc2:=subs(X=1,rhs(Eq2));
```

$$bc2 := \_C1\, e^{(-N)} + \_C2\, e^{N}$$

```
> const:=solve({bc1=0,bc2=1},{_C1,_C2});
```

$$const := \{\_C2 = \frac{e^N}{1+(e^N)^2}, \_C1 = \frac{e^N}{1+(e^N)^2}\}$$

```
> assign(const):
> Eq4:=simplify(convert(Eq2,trig));
```

$$Eq4 := \theta_1(X) = \frac{\cosh(N\,X)}{\cosh(N)}$$

```
> Eq5:=diff(F(X),X,X)-(N^2+I*B)*F(X)=0;
```

$$Eq5 := \left(\frac{d^2}{dX^2}F(X)\right) - (N^2 + B\,I)\,F(X) = 0$$

```
> Eq6:=(dsolve(Eq5,F(X)));
```

$$Eq6 := F(X) = \_C3 \sin(\sqrt{-N^2 - B\,I}\,X) + \_C4 \cos(\sqrt{-N^2 - B\,I}\,X)$$

```
> Eq7:=diff(Eq6,X);
```

$$Eq7 := \frac{d}{dX}F(X) = \_C3 \cos(\sqrt{-N^2 - B\,I}\,X)\sqrt{-N^2 - B\,I} - \_C4 \sin(\sqrt{-N^2 - B\,I}\,X)\sqrt{-N^2 - B\,I}$$

```
> bc3:=simplify(subs(X=0,rhs(Eq7)));
```

$$bc3 := \sqrt{-N^2 - B\,I}\,\_C3$$

```
> bc4:=subs(X=1,rhs(Eq6));
```

$$bc4 := \_C3 \sin(\sqrt{-N^2 - B\,I}) + \_C4 \cos(\sqrt{-N^2 - B\,I})$$

```
> consts:=solve({bc3=0,bc4=1},{_C3,_C4});
```

$$consts := \{\_C3 = 0, \_C4 = \frac{1}{\cos(\sqrt{-N^2 - B\,I})}\}$$

```
> assign(consts):
> Eq8:=simplify(Eq6);
```

$$Eq8 := \mathrm{F}(X) = \frac{\cos(\sqrt{-N^2 - B\,I}\,X)}{\cos(\sqrt{-N^2 - B\,I})}$$

```
> phi:=A*exp(I*B*tau)*rhs(Eq8);
```

$$\phi := \frac{A\,\mathrm{e}^{(B\,\tau\,I)}\cos(\sqrt{-N^2 - B\,I}\,X)}{\cos(\sqrt{-N^2 - B\,I})}$$

```
> assume(B>0,N>0):interface(showassumed=0):
> theta[2]:=evalc(Re(phi));
```

$$\theta_2 := \left(A\cos(B\,\tau)\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right) \middle/ \left(\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2 + \sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\right)\right) + A\sin(B\,\tau)\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right) \middle/ \left(\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2 + \sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\right)$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}\,X}{2}\right)\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}\,X}{2}\right)-\left(A\sin(B\,\tau)\right.$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\Bigg/\left(\right.$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2$$

$$\left.+\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\right)-A\cos(B\,\tau)$$

$$\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\Bigg/\left(\right.$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2$$

$$\left.\left.+\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\right)\right)$$

$$\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}\,X}{2}\right)\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}\,X}{2}\right)$$

```
> theta:=rhs(Eq4)+theta[2];
```

$$\theta := \frac{\cosh(N\,X)}{\cosh(N)}+\left(A\cos(B\,\tau)\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\right.$$

$$\Bigg/\left(\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\right.$$

$$\left.+\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\right)+A\sin(B\,\tau)$$

$$\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\Bigg/\left(\right.$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 \cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2$$

$$+\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 \sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\Bigg)\Bigg)$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}\,X}{2}\right) \cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}\,X}{2}\right) - \Bigg(A\sin(B\,\tau)$$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right) \cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right) \Bigg/ \Bigg($$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 \cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2$$

$$+\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 \sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\Bigg) - A\cos(B\,\tau)$$

$$\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right) \sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right) \Bigg/ \Bigg($$

$$\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 \cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2$$

$$+\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 \sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2\Bigg)\Bigg)$$

$$\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}\,X}{2}\right) \sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}\,X}{2}\right)$$

```
> Eq9:=diff(theta,X):
> Q:=simplify(subs(X=1,Eq9));
```

$$Q := \frac{1}{2}\Bigg(2\sinh(N)\,N\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2 - 2\sinh(N)\,N + 2\sinh(N)\,N\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2 - A\sqrt{2\sqrt{N^4+B^2}-2N^2}\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\cosh(N)\sin(B\,\tau)\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right) - A\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\sqrt{2\sqrt{N^4+B^2}+2N^2}\cosh(N)\sin(B\,\tau)\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right) - A\cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)\sqrt{2\sqrt{N^4+B^2}-2N^2}\cosh(N)\cos(B\,\tau)\sin\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right) + A\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\sqrt{2\sqrt{N^4+B^2}+2N^2}\cosh(N)\cos(B\,\tau)\sinh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)\Bigg) \Bigg/ \left(\cosh(N)\left(\cosh\left(\frac{\sqrt{2\sqrt{N^4+B^2}+2N^2}}{2}\right)^2 - 1 + \cos\left(\frac{\sqrt{2\sqrt{N^4+B^2}-2N^2}}{2}\right)^2\right)\right)$$

```
>theta[t]:=simplify(subs(X=0,A=0.1,B=2*Pi,N=0.1,theta));
```

$$\theta_t := 0.9950207490 - 0.007326874712\cos(6.283185308\,\tau) + 0.03407830800\sin(6.283185308\,\tau)$$

```
> theta[b]:=simplify(1+0.1*cos(2*Pi*tau));
```

$$\theta_b := 1. + 0.1000000000\cos(6.283185308\,\tau)$$

```
> Q[b]:=simplify(subs(A=0.1,N=0.1,B=2*Pi,Q));
```

$$Q_b := 0.009966799454 - 0.1825010309\sin(6.283185308\,\tau) + 0.1911945823\cos(6.283185308\,\tau)$$

```
> plot([theta[t],theta[b]],tau=0..1,legend=["theta[b]",
"theta[t]"],color=black,linestyle=[SOLID,DASH],labels=["tau"
,"theta"]);
```

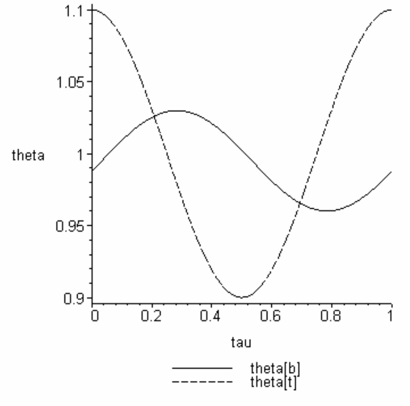

Fig 8.19: Temperature versus time

```
>plot(Q[b],tau=0..1,color=black,labels=["tau","Q[b]"]);
```

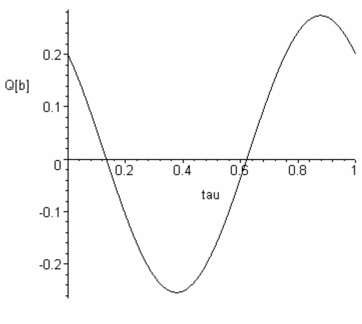

Fig 8.20: Heat flow versus time

```
> for tau from 0 by 0.1 to 1 do
> print(tau,theta[t],theta[b],Q[b]);od;
```

0, 0.9876938743 , 1.100000000 , 0.2011613818
0.1, 1.009123910 , 1.080901699 , 0.05737505125
0.2, 1.025167017 , 1.030901699 , -0.1045196202
0.3, 1.029695275 , 0.9690983006 , -0.2226843703
0.4, 1.020979042 , 0.9190983006 , -0.2519842814
0.5, 1.002347624 , 0.9000000000 , -0.1812277828
0.6, 0.9809175881 , 0.9190983006 , -0.0374414521
0.7, 0.9648744809 , 0.9690983007 , 0.1244532192
0.8, 0.9603462233 , 1.030901699 , 0.2426179694
0.9, 0.9690624560 , 1.080901699 , 0.2719178803
1.0, 0.9876938743 , 1.100000000 , 0.2011613816

The numerical results and the graphical display (Fig 8.19) for A = 0.1, N = 0.1, and B = 2π show that the temperature at the tip, θ_t, lags behind the base temperature θ_b. As expected, the amplitude of oscillation of θ_t is smaller than the amplitude of oscillation of θ_b. The results for the base heat flux, Q_b, shown in Fig 8.20 indicate that the minimum heat flux occurs at $\tau = 0.4$ but the minimum of θ_b occurs at $\tau = 0.5$, that is, the base heat flux leads the base temperature. It is also interesting to observe that Q_b is negative over a portion of the cycle which means that heat is flowing back into the fin over that time period. A full discussion of how the parameters A, B, and N affect the steady periodic performance of the fin can be found in Yang [12] and Aziz and Lunardini [13].

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