where δ_i denotes the roundoff error associated with u_i. Using methods similar to those in the proof of Theorem 5.9, we can produce an error bound for the finite-digit approximations to y_i given by Euler's method.

Theorem 5.10 Let $y(t)$ denote the unique solution to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha \tag{5.12}$$

and $u_0, u_1, \ldots, u_N$ be the approximations obtained using (5.11). If $|\delta_i| < \delta$ for each $i = 0, 1, \ldots, N$ and the hypotheses of Theorem 5.9 hold for (5.12), then

$$|y(t_i) - u_i| \le \frac{1}{L}\left(\frac{hM}{2} + \frac{\delta}{h}\right)[e^{L(t_i-a)} - 1] + |\delta_0|e^{L(t_i-a)}, \tag{5.13}$$

for each $i = 0, 1, \ldots, N$. ■

The error bound (5.13) is no longer linear in h. In fact, since

$$\lim_{h\to 0}\left(\frac{hM}{2} + \frac{\delta}{h}\right) = \infty,$$

the error would be expected to become large for sufficiently small values of h. Calculus can be used to determine a lower bound for the step size h. Letting $E(h) = (hM/2) + (\delta/h)$ implies that $E'(h) = (M/2) - (\delta/h^2)$.

If $h < \sqrt{2\delta/M}$, then $E'(h) < 0$ and $E(h)$ is decreasing.

If $h > \sqrt{2\delta/M}$, then $E'(h) > 0$ and $E(h)$ is increasing.

The minimal value of $E(h)$ occurs when

$$h = \sqrt{\frac{2\delta}{M}}. \tag{5.14}$$

Decreasing h beyond this value tends to increase the total error in the approximation. Normally, however, the value of δ is sufficiently small that this lower bound for h does not affect the operation of Euler's method.

EXERCISE SET 5.2

1. Use Euler's method to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0,$ with $h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1,$ with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2,$ with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1,$ with $h = 0.25$
2. The actual solutions to the initial-value problems in Exercise 1 are given here. Compare the actual error at each step to the error bound.

a. $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$

b. $y(t) = t + \frac{1}{1-t}$

c. $y(t) = t \ln t + 2t$

d. $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$

3. Use Euler's method to approximate the solutions for each of the following initial-value problems.

a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 2, \quad y(1) = 1,$ with $h = 0.1$

b. $y' = 1 + y/t + (y/t)^2, \quad 1 \le t \le 3, \quad y(1) = 0,$ with $h = 0.2$

c. $y' = -(y+1)(y+3), \quad 0 \le t \le 2, \quad y(0) = -2,$ with $h = 0.2$

d. $y' = -5y + 5t^2 + 2t, \quad 0 \le t \le 1, \quad y(0) = \frac{1}{3},$ with $h = 0.1$

4. The actual solutions to the initial-value problems in Exercise 3 are given here. Compute the actual error in the approximations of Exercise 3.

a. $y(t) = \frac{t}{1 + \ln t}$

b. $y(t) = t \tan(\ln t)$

c. $y(t) = -3 + \frac{2}{1 + e^{-2t}}$

d. $y(t) = t^2 + \frac{1}{3}e^{-5t}$

5. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

with exact solution $y(t) = t^2(e^t - e)$:

a. Use Euler's method with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$

c. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.1$, using Eq. (5.10).

6. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

a. Use Euler's method with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$

c. Compute the value of h necessary for $|y(t_i) - w_i| \le 0.05$ using eq. (5.10).

7. Given the initial-value problem

$$y' = -y + t + 1, \quad 0 \le t \le 5, \quad y(0) = 1,$$

with exact solution $y(t) = e^{-t} + t$:

a. Approximate $y(5)$ using Euler's method with $h = 0.2$, $h = 0.1$, and $h = 0.05$.

b. Determine the optimal value of h to use in computing $y(5)$, assuming $\delta = 10^{-6}$ and that Eq. (5.14) is valid.

8. Use the results of Exercise 3 and linear interpolation to approximate the following values of $y(t)$. Compare the approximations obtained to the actual values obtained using the functions given in Exercise 4.

   a. $y(1.25)$ and $y(1.93)$ **b.** $y(2.1)$ and $y(2.75)$

   c. $y(1.4)$ and $y(1.93)$ **d.** $y(0.54)$ and $y(0.94)$

9. Let $E(h) = \frac{hM}{2} + \frac{\delta}{h}$.

   a. For the initial-value problem

$$y' = -y + 1, \quad 0 \le t \le 1, \quad y(0) = 0,$$

   compute the value of h to minimize $E(h)$. Assume $\delta = 5 \times 10^{-(n+1)}$ if you will be using n-digit arithmetic in part (c).

   b. For the optimal h computed in part (a), use Eq. (5.13) to compute the minimal error obtainable.

   c. Compare the actual error obtained using $h = 0.1$ and $h = 0.01$ to the minimal error in part (b). Can you explain the results?

10. Consider the initial-value problem

$$y' = -10y, \quad 0 \le t \le 2, \quad y(0) = 1,$$

   which has solution $y(t) = e^{-10t}$. What happens when Euler's method is applied to this problem with $h = 0.1$? Does this behavior violate Theorem 5.9?

11. In a book entitled *Looking at History Through Mathematics*, Rashevsky [Ra, pp. 103–110] considers a model for a problem involving the production of nonconformists in society. Suppose that a society has a population of $x(t)$ individuals at time t, in years, and that all nonconformists who mate with other nonconformists have offspring who are also nonconformists, while a fixed proportion r of all other offspring are also nonconformist. If the birth and death rates for all individuals are assumed to be the constants b and d, respectively, and if conformists and nonconformists mate at random, the problem can be expressed by the differential equations

$$\frac{dx(t)}{dt} = (b-d)x(t) \quad \text{and} \quad \frac{dx_n(t)}{dt} = (b-d)x_n(t) + rb(x(t) - x_n(t)),$$

   where $x_n(t)$ denotes the number of nonconformists in the population at time t.

   a. Suppose the variable $p(t) = x_n(t)/x(t)$ is introduced to represent the proportion of nonconformists in the society at time t. Show that these equations can be combined and simplified to the single differential equation

$$\frac{dp(t)}{dt} = rb(1 - p(t)).$$

   b. Assuming that $p(0) = 0.01$, $b = 0.02$, $d = 0.015$, and $r = 0.1$, approximate the solution $p(t)$ from $t = 0$ to $t = 50$ when the step size is $h = 1$ year.

   c. Solve the differential equation for $p(t)$ exactly, and compare your result in part (b) when $t = 50$ with the exact value at that time.

12. In a circuit with impressed voltage $\mathcal{E}$ having resistance R, inductance L, and capacitance C in parallel, the current i satisfies the differential equation

$$\frac{di}{dt} = C\frac{d^2\mathcal{E}}{dt^2} + \frac{1}{R}\frac{d\mathcal{E}}{dt} + \frac{1}{L}\mathcal{E}.$$

Suppose $C = 0.3$ farads, $R = 1.4$ ohms, $L = 1.7$ henries, and the voltage is given by

$$\mathcal{E}(t) = e^{-0.06\pi t} \sin(2t - \pi).$$

If $i(0) = 0$, find the current i for the values $t = 0.1j$, where $j = 0, 1, \ldots, 100$.

5.3 Higher-Order Taylor Methods

Since the object of numerical techniques is to determine accurate approximations with minimal effort, we need a means for comparing the efficiency of various approximation methods. The first device we consider is called the *local truncation error* of the method. The local truncation error at a specified step measures the amount by which the exact solution to the differential equation fails to satisfy the difference equation being used for the approximation.

Definition 5.11 The difference method

$$w_0 = \alpha$$
$$w_{i+1} = w_i + h\phi(t_i, w_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

has **local truncation error**

$$\tau_{i+1}(h) = \frac{y_{i+1} - (y_i + h\phi(t_i, y_i))}{h} = \frac{y_{i+1} - y_i}{h} - \phi(t_i, y_i),$$

for each $i = 0, 1, \ldots, N-1$. ■

For Euler's method, the local truncation error at the ith step for the problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - f(t_i, y_i), \quad \text{for each } i = 0, 1, \ldots, N-1,$$

where, as usual, $y_i = y(t_i)$ denotes the exact value of the solution at t_i. This error is a *local error* because it measures the accuracy of the method at a specific step, assuming that the method was exact at the previous step. As such, it depends on the differential equation, the step size, and the particular step in the approximation.

By considering Eq. (5.7) in the previous section, we see that Euler's method has

$$\tau_{i+1}(h) = \frac{h}{2} y''(\xi_i), \quad \text{for some } \xi_i \text{ in } (t_i, t_{i+1}).$$

When $y''(t)$ is known to be bounded by a constant M on $[a, b]$, this implies

Proof Note that Eq. (5.16) can be rewritten

$$y_{i+1} - y_i - hf(t_i, y_i) - \frac{h^2}{2} f'(t_i, y_i) - \cdots - \frac{h^n}{n!} f^{(n-1)}(t_i, y_i) = \frac{h^{n+1}}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)),$$

for some ξ_i in (t_i, t_{i+1}). So the local truncation error is

$$\tau_{i+1}(h) = \frac{y_{i+1} - y_i}{h} - T^{(n)}(t_i, y_i) = \frac{h^n}{(n+1)!} f^{(n)}(\xi_i, y(\xi_i)),$$

for each $i = 0, 1, \ldots, N-1$. Since $y \in C^{n+1}[a, b]$, we have $y^{(n+1)}(t) = f^{(n)}(t, y(t))$ bounded on $[a, b]$ and $\tau_i = O(h^n)$, for each $i = 1, 2, \ldots, N$. ▪ ▪ ▪

EXERCISE SET 5.3

1. Use Taylor's method of order two to approximate the solutions for each of the following initial-value problems.
   a. $y' = te^{3t} - 2y, \quad 0 \le t \le 1, \quad y(0) = 0$, with $h = 0.5$
   b. $y' = 1 + (t - y)^2, \quad 2 \le t \le 3, \quad y(2) = 1$, with $h = 0.5$
   c. $y' = 1 + y/t, \quad 1 \le t \le 2, \quad y(1) = 2$, with $h = 0.25$
   d. $y' = \cos 2t + \sin 3t, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.25$
2. Repeat Exercise 1 using Taylor's method of order four.
3. Use Taylor's method of order two and four to approximate the solution for each of the following initial-value problems.
   a. $y' = y/t - (y/t)^2, \quad 1 \le t \le 1.2, \quad y(1) = 1$, with $h = 0.1$
   b. $y' = \sin t + e^{-t}, \quad 0 \le t \le 1, \quad y(0) = 0$, with $h = 0.5$
   c. $y' = 1/t(y^2 + y), \quad 1 \le t \le 3, \quad y(1) = -2$, with $h = 0.5$
   d. $y' = -ty + 4t/y, \quad 0 \le t \le 1, \quad y(0) = 1$, with $h = 0.25$
4. Use the Taylor method of order two with $h = 0.1$ to approximate the solution to

$$y' = 1 + t\sin(ty), \quad 0 \le t \le 2, \quad y(0) = 0.$$

5. Given the initial-value problem

$$y' = \frac{2}{t}y + t^2 e^t, \quad 1 \le t \le 2, \quad y(1) = 0,$$

   with exact solution $y(t) = t^2(e^t - e)$:
   a. Use Taylor's method of order two with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.
   b. Use the answers generated in part (a) and linear interpolation to approximate y at the following values, and compare them to the actual values of y.
      i. $y(1.04)$ ii. ' $y(1.55)$ iii. $y(1.97)$

c. Use Taylor's method of order four with $h = 0.1$ to approximate the solution, and compare it with the actual values of y.

d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate y at the following values, and compare them to the actual values of y.

i. $y(1.04)$ ii. $y(1.55)$ iii. $y(1.97)$

6. Given the initial-value problem

$$y' = \frac{1}{t^2} - \frac{y}{t} - y^2, \quad 1 \le t \le 2, \quad y(1) = -1,$$

with exact solution $y(t) = -1/t$:

a. Use Taylor's method of order two with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

b. Use the answers generated in part (a) and linear interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$

c. Use Taylor's method of order four with $h = 0.05$ to approximate the solution, and compare it with the actual values of y.

d. Use the answers generated in part (c) and piecewise cubic Hermite interpolation to approximate the following values of y, and compare them to the actual values.

i. $y(1.052)$ ii. $y(1.555)$ iii. $y(1.978)$

7. A projectile of mass $m = 0.11$ kg shot vertically upward with initial velocity $v(0) = 8$ m/s is slowed due to the force of gravity, $F_g = -mg$, and due to air resistance, $F_r = -kv|v|$, where $g = 9.8$ m/s^2 and $k = 0.002$ kg/m. The differential equation for the velocity v is given by

$$mv' = -mg - kv|v|.$$

a. Find the velocity after $0.1, 0.2, \ldots, 1.0$ s.

b. To the nearest tenth of a second, determine when the projectile reaches its maximum height and begins falling.

5.4 Runge-Kutta Methods

The Taylor methods outlined in the previous section have the desirable property of high-order local truncation error, but the disadvantage of requiring the computation and evaluation of the derivatives of $f(t, y)$. This is a complicated and time-consuming procedure for most problems, so the Taylor methods are seldom used in practice.

Runge-Kutta methods have the high-order local truncation error of the Taylor methods while eliminating the need to compute and evaluate the derivatives of $f(t, y)$. Before presenting the ideas behind their derivation, we need to state Taylor's Theorem in two variables. The proof of this result can be found in any standard book on advanced calculus (see, for example, [Fu, p. 331]).

Theorem 5.13 Suppose that $f(t, y)$ and all its partial derivatives of order less than or equal to $n + 1$ are continuous on $D = \{ (t, y) \mid a \le t \le b, c \le y \le d \}$, and let $(t_0, y_0) \in D$. For every

Euler's method with $h = 0.025$, the Midpoint method with $h = 0.05$, and the Runge-Kutta fourth-order method with $h = 0.1$ are compared at the common mesh points of these methods 0.1, 0.2, 0.3, 0.4, and 0.5. Each of these techniques requires 20 functional evaluations to determine the values listed in Table 5.8 to approximate $y(0.5)$. In this example, the fourth-order method is clearly superior. ■

Table 5.8

t_i	Exact	Euler $h = 0.025$	Modified Euler $h = 0.05$	Runge-Kutta Order Four $h = 0.1$
0.0	0.5000000	0.5000000	0.5000000	0.5000000
0.1	0.6574145	0.6554982	0.6573085	0.6574144
0.2	0.8292986	0.8253385	0.8290778	0.8292983
0.3	1.0150706	1.0089334	1.0147254	1.0150701
0.4	1.2140877	1.2056345	1.2136079	1.2140869
0.5	1.4256394	1.4147264	1.4250141	1.4256384

EXERCISE SET 5.4

1. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.
   a. $y' = te^{3t} - 2y$, $0 \le t \le 1$, $y(0) = 0$, with $h = 0.5$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.
   b. $y' = 1+(t-y)^2$, $2 \le t \le 3$, $y(2) = 1$, with $h = 0.5$; actual solution $y(t) = t + \frac{1}{1-t}$.
   c. $y' = 1 + y/t$, $1 \le t \le 2$, $y(1) = 2$, with $h = 0.25$; actual solution $y(t) = t\ln t + 2t$.
   d. $y' = \cos 2t + \sin 3t$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.25$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.
2. Repeat Exercise 1 using Heun's method.
3. Repeat Exercise 1 using the Midpoint method.
4. Use the Modified Euler method to approximate the solutions to each of the following initial-value problems, and compare the results to the actual values.
   a. $y' = y/t - (y/t)^2$, $1 \le t \le 2$, $y(1) = 1$, with $h = 0.1$; actual solution $y(t) = t/(1 + \ln t)$.
   b. $y' = 1 + y/t + (y/t)^2$, $1 \le t \le 3$, $y(1) = 0$, with $h = 0.2$; actual solution $y(t) = t\tan(\ln t)$.
   c. $y' = -(y + 1)(y + 3)$, $0 \le t \le 2$, $y(0) = -2$, with $h = 0.2$; actual solution $y(t) = -3 + 2(1 + e^{-2t})^{-1}$.
   d. $y' = -5y + 5t^2 + 2t$, $0 \le t \le 1$, $y(0) = \frac{1}{3}$, with $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$.
5. Use the results of Exercise 4 and linear interpolation to approximate values of $y(t)$, and compare the results to the actual values.
   a. $y(1.25)$ and $y(1.93)$
   b. $y(2.1)$ and $y(2.75)$
   c. $y(1.3)$ and $y(1.93)$
   d. $y(0.54)$ and $y(0.94)$

6. Repeat Exercise 4 using Heun's method.
7. Repeat Exercise 5 using the results of Exercise 6.
8. Repeat Exercise 4 using the Midpoint method.
9. Repeat Exercise 5 using the results of Exercise 8.
10. Repeat Exercise 1 using the Runge-Kutta method of order four.
11. Repeat Exercise 4 using the Runge-Kutta method of order four.
12. Use the results of Exercise 11 and Cubic Hermite interpolation to approximate values of $y(t)$, and compare the approximations to the actual values.
    a. $y(1.25)$ and $y(1.93)$
    b. $y(2.1)$ and $y(2.75)$
    c. $y(1.3)$ and $y(1.93)$
    d. $y(0.54)$ and $y(0.94)$
13. Show that the Midpoint method, the Modified Euler method, and Heun's method give the same approximations to the initial-value problem

$$y' = -y + t + 1, \quad 0 \leq t \leq 1, \quad y(0) = 1,$$

for any choice of h. Why is this true?

14. Water flows from an inverted conical tank with circular orifice at the rate

$$\frac{dx}{dt} = -0.6\pi r^2 \sqrt{2g}\frac{\sqrt{x}}{A(x)},$$

where r is the radius of the orifice, x is the height of the liquid level from the vertex of the cone, and $A(x)$ is the area of the cross section of the tank x units above the orifice. Suppose $r = 0.1$ ft, $g = 32.1$ ft/s^2, and the tank has an initial water level of 8 ft and initial volume of $512(\pi/3)$ ft^3.
    a. Compute the water level after 10 min with $h = 20$ s.
    b. Determine, to within 1 min, when the tank will be empty.
15. The irreversible chemical reaction in which two molecules of solid potassium dichromate ($K_2Cr_2O_7$), two molecules of water (H_2O), and three atoms of solid sulfur (S) combine to yield three molecules of the gas sulfur dioxide (SO_2), four molecules of solid potassium hydroxide (KOH), and two molecules of solid chromic oxide (Cr_2O_3) can be represented symbolically by the *stoichiometric equation*:

$$2K_2Cr_2O_7 + 2H_2O + 3S \longrightarrow 4KOH + 2Cr_2O_3 + 3SO_2.$$

If n_1 molecules of $K_2Cr_2O_7$, n_2 molecules of H_2O, and n_3 molecules of S are originally available, the following differential equation describes the amount $x(t)$ of KOH after time t:

$$\frac{dx}{dt} = k\left(n_1 - \frac{x}{2}\right)^2\left(n_2 - \frac{x}{2}\right)^2\left(n_3 - \frac{3x}{4}\right)^3,$$

where k is the velocity constant of the reaction. If $k = 6.22 \times 10^{-19}$, $n_1 = n_2 = 2 \times 10^3$, and $n_3 = 3 \times 10^3$, how many units of potassium hydroxide will have been formed after 0.2 s?

16. Show that the difference method

$$w_0 = \alpha,$$

$$w_{i+1} = w_i + a_1 f(t_i, w_i) + a_2 f(t_i + \alpha_2, w_1 + \delta_2 f(t_i, w_i)),$$

for each $i = 0, 1, \ldots, N-1$, cannot have local truncation error $O(h^3)$ for any choice of constants a_1, a_2, α_2, and δ_2.

17. The Runge-Kutta method of order four can be written in the form

$$w_0 = \alpha,$$

$$\begin{aligned} w_{i+1} = w_i &+ \frac{h}{6} f(t_i, w_i) + \frac{h}{3} f(t_i + \alpha_1 h, w_i + \delta_1 h f(t_i, w_i)) \\ &+ \frac{h}{3} f(t_i + \alpha_2 h, w_i + \delta_2 h f(t_i + \gamma_2 h, w_i + \gamma_3 h f(t_i, w_i))) \\ &+ \frac{h}{6} f(t_i + \alpha_3 h, w_i + \delta_3 h f(t_i + \gamma_4 h, w_i + \gamma_5 h f(t_i + \gamma_6 h, w_i + \gamma_7 h f(t_i, w_i)))). \end{aligned}$$

Find the values of the constants

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \delta_1,\ \delta_2,\ \delta_3,\ \gamma_2,\ \gamma_3,\ \gamma_4,\ \gamma_5,\ \gamma_6, \text{ and } \gamma_7.$$

5.5 Error Control and the Runge-Kutta-Fehlberg Method

The appropriate use of varying step size was seen in Section 4.6 to produce computationally efficient integral approximating methods. In itself, this might not be sufficient to favor these methods due to the increased complication of applying them. However, they have another feature that makes them worthwhile. They incorporate in the step-size procedure an estimate of the truncation error that does not require the approximation of the higher derivatives of the function. These methods are called *adaptive* because they adapt the number and position of the nodes used in the approximation to ensure that the truncation error is kept within a specified bound.

There is a close connection between the problem of approximating the value of a definite integral and that of approximating the solution to an initial-value problem. It is not surprising, then, that there are adaptive methods for approximating the solutions to initial-value problems and that these methods are not only efficient, but also incorporate the control of error.

An ideal difference-equation method

$$w_{i+1} = w_i + h_i \phi(t_i, w_i, h_i), \quad i = 0, 1, \ldots, N-1,$$

for approximating the solution, $y(t)$, to the initial-value problem

$$y' = f(t, y), \quad a \le t \le b, \quad y(a) = \alpha,$$

would have the property that, given a tolerance $\varepsilon > 0$, the minimal number of mesh points would be used to ensure that the global error, $|y(t_i) - w_i|$, would not exceed ε for any $i = 0, 1, \ldots, N$. Having a minimal number of mesh points and also controlling the global error of a difference method is, not surprisingly, inconsistent with the points being equally spaced in the interval. In this section we examine techniques used to control the error of a difference-equation method in an efficient manner by the appropriate choice of mesh points.

Although we cannot generally determine the global error of a method, we will see in Section 5.10 that there is a close connection between the local truncation error and the

The local truncation error involved with a predictor-corrector method of the Milne-Simpson type is generally smaller than that of the Adams-Bashforth-Moulton method. But the technique has limited use because of roundoff error problems, which do not occur with the Adams procedure. Elaboration on this difficulty is given in Section 5.10.

EXERCISE SET 5.6

1. Use all the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use exact starting values, and compare the results to the actual values.
   a. $y' = te^{3t} - 2y$, $0 \le t \le 1$, $y(0) = 0$, with $h = 0.2$; actual solution $y(t) = \frac{1}{5}te^{3t} - \frac{1}{25}e^{3t} + \frac{1}{25}e^{-2t}$.
   b. $y' = 1+(t-y)^2$, $2 \le t \le 3$, $y(2) = 1$, with $h = 0.2$; actual solution $y(t) = t + \frac{1}{1-t}$.
   c. $y' = 1+y/t$, $1 \le t \le 2$, $y(1) = 2$, with $h = 0.2$; actual solution $y(t) = t\ln t + 2t$.
   d. $y' = \cos 2t + \sin 3t$, $0 \le t \le 1$, $y(0) = 1$, with $h = 0.2$; actual solution $y(t) = \frac{1}{2}\sin 2t - \frac{1}{3}\cos 3t + \frac{4}{3}$.
2. Use all the Adams-Moulton methods to approximate the solutions to the Exercises 1(a), 1(c), and 1(d). In each case use exact starting values, and explicitly solve for w_{i+1}. Compare the results to the actual values.
3. Use each of the Adams-Bashforth methods to approximate the solutions to the following initial-value problems. In each case use starting values obtained from the Runge-Kutta method of order four. Compare the results to the actual values.
   a. $y' = y/t-(y/t)^2$, $1 \le t \le 2$, $y(1) = 1$, with $h = 0.1$; actual solution $y(t) = \frac{t}{1+\ln t}$.
   b. $y' = 1 + y/t + (y/t)^2$, $1 \le t \le 3$, $y(1) = 0$, with $h = 0.2$; actual solution $y(t) = t\tan(\ln t)$.
   c. $y' = -(y+1)(y+3)$, $0 \le t \le 2$, $y(0) = -2$, with $h = 0.1$; actual solution $y(t) = -3 + 2/(1+e^{-2t})$.
   d. $y' = -5y + 5t^2 + 2t$, $0 \le t \le 1$, $y(0) = 1/3$, with $h = 0.1$; actual solution $y(t) = t^2 + \frac{1}{3}e^{-5t}$.
4. Use Algorithm 5.4 to approximate the solutions to the initial-value problems in Exercise 1.
5. Use Algorithm 5.4 to approximate the solutions to the initial-value problems in Exercise 3.
6. Change Algorithm 5.4 so that the corrector can be iterated for a given number p iterations. Repeat Exercise 5 with $p = 2$, 3, and 4 iterations. Which choice of p gives the best answer for each initial-value problem?
7. The initial-value problem

$$y' = e^y, \quad 0 \le t \le 0.20, \quad y(0) = 1$$

has solution

$$y(t) = 1 - \ln(1 - et).$$

Applying the three-step Adams-Moulton method to this problem is equivalent to finding the fixed point w_{i+1} of

$$g(w) = w_i + \frac{h}{24}[9e^w + 19e^{w_i} - 5e^{w_{i-1}} + e^{w_{i-2}}].$$

a. With $h = 0.01$, obtain w_{i+1} by functional iteration for $i = 2, \ldots, 19$ using exact starting values w_0, w_1, and w_2. At each step use w_i to initially approximate w_{i+1}.

b. Will Newton's method speed the convergence over functional iteration?

8. Use the Milne-Simpson Predictor-Corrector method to approximate the solutions to the initial-value problems in Exercise 3.

9. **a.** Derive Eq. (5.32) by using the Lagrange form of the interpolating polynomial.

b. Derive Eq. (5.34) by using Newton's backward-difference form of the interpolating polynomial.

10. Derive Eq. (5.33) by the following method. Set

$$y(t_{i+1}) = y(t_i) + ahf(t_i, y(t_i)) + bhf(t_{i-1}, y(t_{i-1})) + chf(t_{i-2}, y(t_{i-2})).$$

Expand $y(t_{i+1})$, $f(t_{i-2}, y(t_{i-2}))$, and $f(t_{i-1}, y(t_{i-1}))$ in Taylor series about $(t_i, y(t_i))$, and equate the coefficients of h, h^2 and h^3 to obtain a, b, and c.

11. Derive Eq. (5.36) and its local truncation error by using an appropriate form of an interpolating polynomial.

12. Derive Simpson's method by applying Simpson's rule to the integral

$$y(t_{i+1}) - y(t_{i-1}) = \int_{t_{i-1}}^{t_{i+1}} f(t, y(t))\, dt.$$

13. Derive Milne's method by applying the open Newton-Cotes formula (4.29) to the integral

$$y(t_{i+1}) - y(t_{i-3}) = \int_{t_{i-3}}^{t_{i+1}} f(t, y(t))\, dt.$$

14. Verify the entries in Table 5.10.

5.7 Variable Step-Size Multistep Methods

The Runge-Kutta-Fehlberg method is used for error control because at each step it provides, at little additional cost, *two* approximations that can be compared and related to the local error. Predictor-corrector techniques always generate two approximations at each step, so they are natural candidates for error-control adaptation.

To demonstrate the error-control procedure, we will construct a variable step-size predictor-corrector method using the four-step explicit Adams-Bashforth method as predictor and the three-step implicit Adams-Moulton method as corrector.

The Adams-Bashforth four-step method comes from the relation

$$y(t_{i+1}) = y(t_i) + \frac{h}{24}[55f(t_i, y(t_i)) - 59f(t_{i-1}, y(t_{i-1})) + 37f(t_{i-2}, y(t_{i-2})) - 9f(t_{i-3}, y(t_{i-3}))] + \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^5,$$

for some $\hat{\mu}_i \in (t_{i-3}, t_{i+1})$. The assumption that the approximations $w_0, w_1, \ldots, w_i$ are all exact implies that the Adams-Bashforth truncation error is

$$\frac{y(t_{i+1}) - w_{i+1}^{(0)}}{h} = \frac{251}{720}y^{(5)}(\hat{\mu}_i)h^4. \tag{5.39}$$

We can approximate $y(1.0)$ using the command

```
>g(1.0);
```

to give

$$\left[t = 1.0, y(t) = -.353394346807534676, \frac{\partial}{\partial t}y(t) = 2.57874665940482072\right]$$

The other one-step methods can be extended to systems in a similar way. When error control methods like the Runge-Kutta-Fehlberg method are extended, each component of the numerical solution $(w_{1j}, w_{2j}, \ldots, w_{mj})$ must be examined for accuracy. If any of the components fail to be sufficiently accurate, the entire numerical solution $(w_{1j}, w_{2j}, \ldots, w_{mj})$ must be recomputed.

The multistep methods and predictor-corrector techniques can also be extended to systems. Again, if error control is used, each component must be accurate. The extension of the extrapolation technique to systems can also be done, but the notation becomes quite involved. If this topic is of interest, see [HNW1].

Convergence theorems and error estimates for systems are similar to those considered in Section 5.10 for the single equations, except that the bounds are given in terms of vector norms, a topic considered in Chapter 7. (A good reference for these theorems is [Ge1, pp. 45–72].)

EXERCISE SET 5.9

1. Use the Runge-Kutta method for systems to approximate the solutions of the following systems of first-order differential equations, and compare the results to the actual solutions.
   a. $u_1' = 3u_1 + 2u_2 - (2t^2 + 1)e^{2t}, \quad 0 \le t \le 1, \quad u_1(0) = 1;$
   $u_2' = 4u_1 + u_2 + (t^2 + 2t - 4)e^{2t}, \quad 0 \le t \le 1, \quad u_2(0) = 1;$
   $h = 0.2;$ actual solutions $u_1(t) = \frac{1}{3}e^{5t} - \frac{1}{3}e^{-t} + e^{2t}$ and $u_2(t) = \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t} + t^2e^{2t}$.
   b. $u_1' = -4u_1 - 2u_2 + \cos t + 4\sin t, \quad 0 \le t \le 2, \quad u_1(0) = 0;$
   $u_2' = 3u_1 + u_2 - 3\sin t, \quad 0 \le t \le 2, \quad u_2(0) = -1;$
   $h = 0.1;$ actual solutions $u_1(t) = 2e^{-t} - 2e^{-2t} + \sin t$ and $u_2(t) = -3e^{-t} + 2e^{-2t}$.
   c. $u_1' = u_2, \quad 0 \le t \le 2, \quad u_1(0) = 1;$
   $u_2' = -u_1 - 2e^t + 1, \quad 0 \le t \le 2, \quad u_2(0) = 0;$
   $u_3' = -u_1 - e^t + 1, \quad 0 \le t \le 2, \quad u_3(0) = 1;$
   $h = 0.5;$ actual solutions $u_1(t) = \cos t + \sin t - e^t + 1$, $u_2(t) = -\sin t + \cos t - e^t$, and $u_3(t) = -\sin t + \cos t$.
   d. $u_1' = u_2 - u_3 + t, \quad 0 \le t \le 1, \quad u_1(0) = 1;$
   $u_2' = 3t^2, \quad 0 \le t \le 1, \quad u_2(0) = 1;$
   $u_3' = u_2 + e^{-t}, \quad 0 \le t \le 1, \quad u_3(0) = -1;$
   $h = 0.1;$ actual solutions $u_1(t) = -0.05t^5 + 0.25t^4 + t + 2 - e^{-t}$, $u_2(t) = t^3 + 1$, and $u_3(t) = 0.25t^4 + t - e^{-t}$.
2. Use the Runge-Kutta for Systems Algorithm to approximate the solutions of the following higher-order differential equations, and compare the results to the actual solutions.
   a. $y'' - 2y' + y = te^t - t, \quad 0 \le t \le 1, \quad y(0) = y'(0) = 0,$ with $h = 0.1;$ actual solution $y(t) = \frac{1}{6}t^3e^t - te^t + 2e^t - t - 2$.