بست کنترل : با پیروزی مربوطم

EXERCISE SET 6.1

1. For each of the following linear systems, obtain a solution by graphical methods, if possible. Explain the results from a geometrical standpoint.

   a. $x_1 + 2x_2 = 3,$
   $x_1 - x_2 = 0.$

   b. $x_1 + 2x_2 = 0.$
   $x_1 - x_2 = 0.$

   c. $x_1 + 2x_2 = 3,$
   $2x_1 + 4x_2 = 6.$

   d. $x_1 + 2x_2 = 3.$
   $-2x_1 - 4x_2 = 6.$

   e. $x_1 + 2x_2 = 0,$
   $2x_1 + 4x_2 = 0.$

   f. $2x_1 + x_2 = -1,$
   $x_1 + x_2 = 2,$
   $x_1 - 3x_2 = 5.$

   g. $2x_1 + x_2 = -1,$
   $4x_1 + 2x_2 = -2,$
   $x_1 - 3x_2 = 5.$

   h. $2x_1 + x_2 + x_3 = 1,$
   $2x_1 + 4x_2 - x_3 = -1.$

2. Use Gaussian elimination with backward substitution and two-digit rounding arithmetic to solve the following linear systems. Do not reorder the equations. (The exact solution to each system is $x_1 = 1, x_2 = -1, x_3 = 3$.)

   a. $4x_1 - x_2 + x_3 = 8,$
   $2x_1 + 5x_2 + 2x_3 = 3,$
   $x_1 + 2x_2 + 4x_3 = 11.$

   b. $4x_1 + x_2 + 2x_3 = 9,$
   $2x_1 + 4x_2 - x_3 = -5,$
   $x_1 + x_2 - 3x_3 = -9.$

3. Use the Gaussian Elimination Algorithm to solve the following linear systems, ~~if possible, and determine whether row interchanges are necessary~~:

   a. $x_1 - x_2 + 3x_3 = 2,$
   $3x_1 - 3x_2 + x_3 = -1,$
   $x_1 + x_2 = 3.$

   b. $2x_1 - 1.5x_2 + 3x_3 = 1,$
   $-x_1 + 2x_3 = 3,$
   $4x_1 - 4.5x_2 + 5x_3 = 1.$

   c. $2x_1 = 3,$
   $x_1 + 1.5x_2 = 4.5,$
   $- 3x_2 + 0.5x_3 = -6.6,$
   $2x_1 - 2x_2 + x_3 + x_4 = 0.8.$

   d. $x_1 - \frac{1}{2}x_2 + x_3 = 4,$
   $2x_1 - x_2 - x_3 + x_4 = 5,$
   $x_1 + x_2 = 2,$
   $x_1 - \frac{1}{2}x_2 + x_3 + x_4 = 5.$

   e. $x_1 + x_2 + x_4 = 2,$
   $2x_1 + x_2 - x_3 + x_4 = 1,$
   $4x_1 - x_2 - 2x_3 + 2x_4 = 0,$
   $3x_1 - x_2 - x_3 + 2x_4 = -3.$

   f. $x_1 + x_2 + x_4 = 2,$
   $2x_1 + x_2 - x_3 + x_4 = 1,$
   $-x_1 + 2x_2 + 3x_3 - x_4 = 4,$
   $3x_1 - x_2 - x_3 + 2x_4 = -3.$

4. Use the Gaussian Elimination Algorithm and single-precision arithmetic on a computer to solve the following linear systems.

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a. $$\begin{aligned}\tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 &= 9,\\ \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 &= 8,\\ \tfrac{1}{2}x_1 + x_2 + 2x_3 &= 8.\end{aligned}$$

b. $$\begin{aligned}3.333x_1 + 15920x_2 - 10.333x_3 &= 15913,\\ 2.222x_1 + 16.71x_2 + 9.612x_3 &= 28.544,\\ 1.5611x_1 + 5.1791x_2 + 1.6852x_3 &= 8.4254.\end{aligned}$$

c. $$\begin{aligned}x_1 + \tfrac{1}{2}x_2 + \tfrac{1}{3}x_3 + \tfrac{1}{4}x_4 &= \tfrac{1}{6},\\ \tfrac{1}{2}x_1 + \tfrac{1}{3}x_2 + \tfrac{1}{4}x_3 + \tfrac{1}{5}x_4 &= \tfrac{1}{7},\\ \tfrac{1}{3}x_1 + \tfrac{1}{4}x_2 + \tfrac{1}{5}x_3 + \tfrac{1}{6}x_4 &= \tfrac{1}{8},\\ \tfrac{1}{4}x_1 + \tfrac{1}{5}x_2 + \tfrac{1}{6}x_3 + \tfrac{1}{7}x_4 &= \tfrac{1}{9}.\end{aligned}$$

d. $$\begin{aligned}2x_1 + x_2 - x_3 + x_4 - 3x_5 &= 7,\\ x_1 \qquad + 2x_3 - x_4 + x_5 &= 2,\\ -2x_2 - x_3 + x_4 - x_5 &= -5,\\ 3x_1 + x_2 - 4x_3 \qquad + 5x_5 &= 6,\\ x_1 - x_2 - x_3 - x_4 + x_5 &= 3.\end{aligned}$$

5. Given the linear system

$$\begin{aligned}2x_1 - 6\alpha x_2 &= 3,\\ 3\alpha x_1 - x_2 &= \tfrac{3}{2}.\end{aligned}$$

a. Find value(s) of α for which the system has no solutions.
b. Find value(s) of α for which the system has an infinite number of solutions.
c. Assuming a unique solution exists for a given α, find the solution.

6. Given the linear system

$$\begin{aligned}x_1 - x_2 + \alpha x_3 &= -2,\\ -x_1 + 2x_2 - \alpha x_3 &= 3,\\ \alpha x_1 + x_2 + x_3 &= 2.\end{aligned}$$

a. Find value(s) of α for which the system has no solutions.
b. Find value(s) of α for which the system has an infinite number of solutions.
c. Assuming a unique solution exists for a given α, find the solution.

7. Show that the operations

a. $(\lambda E_i) \to (E_i)$ b. $(E_i + \lambda E_j) \to (E_i)$ c. $(E_i) \leftrightarrow (E_j)$

do not change the solution set of a linear system.

8. **Gauss-Jordan Method:** This method is described as follows. Use the ith equation to eliminate not only x_i from the equations $E_{i+1}, E_{i+2}, \ldots, E_n$, as was done in the Gaussian elimination method, but also from $E_1, E_2, \ldots, E_{i-1}$. Upon reducing $[A, \mathbf{b}]$ to:

$$\left[\begin{array}{cccc:c} a_{11}^{(1)} & 0 & \cdots & 0 & a_{1,n+1}^{(1)}\\ 0 & a_{22}^{(2)} & \ddots & \vdots & a_{2,n+1}^{(2)}\\ \vdots & \ddots & \ddots & 0 & \vdots\\ 0 & \cdots & 0 & a_{nn}^{(n)} & a_{n,n+1}^{(n)}\end{array}\right],$$

so that new scale factors were determined each time a row interchange decision was to be made. In this case, the term $n(n-1)$ in Eq. (6.7) would be replaced by

$$\sum_{k=2}^{n} k(k-1) = \frac{1}{3}n(n^2-1).$$

As a consequence, this pivoting technique would add $O(n^3/3)$ comparisons, in addition to the $[n(n+1)/2]-1$ divisions. If a system warrants this type of pivoting, **complete** (or *maximal*) **pivoting** should instead be used. Complete pivoting at the kth step searches all the entries a_{ij}, for $i = k, k+1, \ldots, n$ and $j = k, k+1, \ldots, n$, to find the entry with the largest magnitude. Both row and column interchanges are performed to bring this entry to the pivot position. The first step of total pivoting requires that n^2-1 comparisons be performed, the second step requires $(n-1)^2-1$ comparisons, and so on. Hence the total additional time required to incorporate complete pivoting into Gaussian elimination is

$$\sum_{k=2}^{n}(k^2-1) = \frac{n(n-1)(2n+5)}{6}$$

comparisons. This figure is comparable to the number required for the modified scaled-column pivoting technique, but no divisions are required. Complete pivoting is, consequently, the strategy recommended for systems where accuracy is essential and the amount of execution time needed for this method can be justified.

EXERCISE SET 6.2

1. Find the row interchanges that are required to solve the following linear systems using Algorithm 6.1.

   a. $$\begin{aligned} x_1 - 5x_2 + x_3 &= 7,\\ 10x_1 + 20x_3 &= 6,\\ 5x_1 - x_3 &= 4.\end{aligned}$$

   b. $$\begin{aligned} x_1 + x_2 - x_3 &= 1,\\ x_1 + x_2 + 4x_3 &= 2,\\ 2x_1 - x_2 + 2x_3 &= 3.\end{aligned}$$

   c. $$\begin{aligned} 2x_1 - 3x_2 + 2x_3 &= 5,\\ -4x_1 + 2x_2 - 6x_3 &= 14,\\ 2x_1 + 2x_2 + 4x_3 &= 8.\end{aligned}$$

   d. $$\begin{aligned} x_2 + x_3 &= 6,\\ x_1 - 2x_2 - x_3 &= 4,\\ x_1 - x_2 + x_3 &= 5.\end{aligned}$$

2. Repeat Exercise 1 using Algorithm 6.2.
3. Repeat Exercise 1 using Algorithm 6.3.
4. Repeat Exercise 1 using complete pivoting.
5. Use Gaussian elimination and three-digit chopping arithmetic to solve the following linear systems, and compare the approximations to the actual solution. without pivoting

   ★ a. $$\begin{aligned} 0.03x_1 + 58.9x_2 &= 59.2,\\ 5.31x_1 - 6.10x_2 &= 47.0.\end{aligned}$$
   Actual solution $(10, 1)^t$.

   b. $$\begin{aligned} 58.9x_1 + 0.03x_2 &= 59.2,\\ -6.10x_1 + 5.31x_2 &= 47.0.\end{aligned}$$
   Actual solution $(1, 10)^t$.

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c. $3.03x_1 - 12.1x_2 + 14x_3 = -119,$
$-3.03x_1 + 12.1x_2 - 7x_3 = 120,$
$6.11x_1 - 14.2x_2 + 21x_3 = -139.$
Actual solution $(0, 10, \frac{1}{7})^t$.

✱d. $3.3330x_1 + 15920x_2 + 10.333x_3 = 7953,$
$2.2220x_1 + 16.710x_2 + 9.6120x_3 = 0.965,$
$-1.5611x_1 + 5.1792x_2 - 1.6855x_3 = 2.714.$
Actual solution $(1, 0.5, -1)^t$.

e. $1.19x_1 + 2.11x_2 - 100x_3 + x_4 = 1.12,$
$14.2x_1 - 0.122x_2 + 12.2x_3 - x_4 = 3.44,$
$100x_2 - 99.9x_3 + x_4 = 2.15,$
$15.3x_1 + 0.110x_2 - 13.1x_3 - x_4 = 4.16.$
Actual solution $(0.17682530, 0.01269269, -0.02065405, -1.18260870)^t$.

f. $\pi x_1 - ex_2 + \sqrt{2}x_3 - \sqrt{3}x_4 = \sqrt{11},$
$\pi^2 x_1 + ex_2 - e^2 x_3 + \frac{3}{7}x_4 = 0,$
$\sqrt{5}x_1 - \sqrt{6}x_2 + x_3 - \sqrt{2}x_4 = \pi,$
$\pi^3 x_1 + e^2 x_2 - \sqrt{7}x_3 + \frac{1}{9}x_4 = \sqrt{2}.$
Actual solution $(0.78839378, -3.12541367, 0.16759660, 4.55700252)^t$.

6. Repeat Exercise 5 using three-digit rounding arithmetic.
✱ 7. Repeat Exercise 5 using Gaussian elimination with partial pivoting. (a, d)
8. Repeat Exercise 6 using Gaussian elimination with partial pivoting.
✱ 9. Repeat Exercise 5 using Gaussian elimination with scaled partial pivoting. (a, d)
10. Repeat Exercise 6 using Gaussian elimination with scaled partial pivoting.
11. Repeat Exercise 5 using Algorithm 6.1 with single-precision computer arithmetic.
12. Repeat Exercise 5 using Algorithm 6.2 with single-precision computer arithmetic.
13. Repeat Exercise 5 using Algorithm 6.3 with single-precision computer arithmetic.
14. Construct an algorithm for the complete pivoting procedure discussed in the text.
15. Use the complete pivoting algorithm developed in Exercise 14 to obtain solutions to
    a. Exercise 5   b. Exercise 6   c. Exercise 11
16. Suppose that

$$2x_1 + x_2 + 3x_3 = 1,$$
$$4x_1 + 6x_2 + 8x_3 = 5,$$
$$6x_1 + \alpha x_2 + 10x_3 = 5,$$

with $|\alpha| < 10$. For which of the following values of α will there be no row interchange required when solving this system using scaled partial pivoting?

a. $\alpha = 6$ b. $\alpha = 9$ c. $\alpha = -3$

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have transposes

$$A^t = \begin{bmatrix} 7 & 3 & 0 \\ 2 & 5 & 5 \\ 0 & -1 & -6 \end{bmatrix}, \quad B^t = \begin{bmatrix} 2 & 3 \\ 4 & -5 \\ 7 & -1 \end{bmatrix}, \quad C^t = \begin{bmatrix} 6 & 4 & -3 \\ 4 & -2 & 0 \\ -3 & 0 & 1 \end{bmatrix}.$$

The matrix C is symmetric since $C^t = C$, but the matrices A and B are not.

The proof of the next result follows directly from the definition of the transpose.

Theorem 6.13 The following operations involving the transpose of a matrix hold whenever the operation is possible:

a. $(A^t)^t = A$, **b.** $(A + B)^t = A^t + B^t$,

c. $(AB)^t = B^t A^t$, **d.** if A^{-1} exists, then $(A^{-1})^t = (A^t)^{-1}$. ∎

Any CAS can be used to perform matrix operations. For example, matrix addition is done in Maple with `matadd(A,B)` or `evalm(A+B)`. Scalar multiplication is defined by `scalarmul(A,c)` or `evalm(c*A)`.

Matrix multiplication is done using `multiply(A,B)` or `evalm(A&*B)`. Matrix transposition is achieved with `transpose(A)` and matrix inversion, with `inverse(A)`.

EXERCISE SET 6.3

1. Determine which of the following matrices are nonsingular, and compute the inverse of these matrices:

a. $\begin{bmatrix} 4 & 2 & 6 \\ 3 & 0 & 7 \\ -2 & -1 & -3 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ **d.** $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & 2 & -4 & -2 \\ 2 & 1 & 1 & 5 \\ -1 & 0 & -2 & -4 \end{bmatrix}$

e. $\begin{bmatrix} 4 & 0 & 0 & 0 \\ 6 & 7 & 0 & 0 \\ 9 & 11 & 1 & 0 \\ 5 & 4 & 1 & 1 \end{bmatrix}$ **f.** $\begin{bmatrix} 2 & 0 & 1 & 2 \\ 1 & 1 & 0 & 2 \\ 2 & -1 & 3 & 1 \\ 3 & -1 & 4 & 3 \end{bmatrix}$

2. Consider the four 3×3 linear systems having the same coefficient matrix:

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 2, \\ x_1 + x_2 - x_3 &= -1, \\ -x_1 + x_2 - 3x_3 &= 0; \end{aligned} \qquad \begin{aligned} 2x_1 - 3x_2 + x_3 &= 6, \\ x_1 + x_2 - x_3 &= 4, \\ -x_1 + x_2 - 3x_3 &= 5; \end{aligned}$$

$$\begin{aligned} 2x_1 - 3x_2 + x_3 &= 0, \\ x_1 + x_2 - x_3 &= 1, \\ -x_1 + x_2 - 3x_3 &= -3; \end{aligned} \qquad \begin{aligned} 2x_1 - 3x_2 + x_3 &= -1, \\ x_1 + x_2 - x_3 &= 0, \\ -x_1 + x_2 - 3x_3 &= 0. \end{aligned}$$

a. Solve the linear systems by applying Gaussian elimination to the augmented matrix

$$\left[\begin{array}{rrr:rrrr} 2 & -3 & 1 & 2 & 6 & 0 & -1 \\ 1 & 1 & -1 & -1 & 4 & 1 & 0 \\ -1 & 1 & -3 & 0 & 5 & -3 & 0 \end{array}\right].$$

b. Solve the linear systems by finding and multiplying by the inverse of

$$A = \begin{bmatrix} 2 & -3 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & -3 \end{bmatrix}.$$

c. Which method requires more operations?

3. Repeat Exercise 2 using the linear systems

$$\begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 6, \\ x_1 - x_3 + x_4 &= 4, \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= -2, \\ x_2 + x_3 - x_4 &= 5; \end{aligned} \qquad \begin{aligned} x_1 - x_2 + 2x_3 - x_4 &= 1, \\ x_1 - x_3 + x_4 &= 1, \\ 2x_1 + x_2 + 3x_3 - 4x_4 &= 2, \\ -x_2 + x_3 - x_4 &= -1. \end{aligned}$$

4. Prove the following statements or provide counterexamples to show they are not true.

a. The product of two symmetric matrices is symmetric.

b. The inverse of a nonsingular symmetric matrix is a nonsingular symmetric matrix.

c. If A and B are $n \times n$ matrices, then $(AB)^t = A^t B^t$.

5. The following statements are needed to prove Theorem 6.11.

a. Show that if A^{-1} exists, it is unique.

b. Show that if A is nonsingular, then $(A^{-1})^{-1} = A$.

c. Show that if A and B are nonsingular $n \times n$ matricies, then $(AB)^{-1} = B^{-1}A^{-1}$.

6. Prove Theorem 6.5.

7. **a.** Show that the product of two $n \times n$ lower triangular matrices is lower triangular.

b. Show that the product of two $n \times n$ upper triangular matrices is upper triangular.

c. Show that the inverse of a nonsingular $n \times n$ lower triangular matrix is lower triangular.

8. Suppose m linear systems

$$A\mathbf{x}^{(p)} = \mathbf{b}^{(p)}, \quad p = 1, 2, \ldots, m,$$

are to be solved, each with the $n \times n$ coefficient matrix A.

a. Show that Gaussian elimination with backward substitution applied to the augmented matrix

$$[A : \mathbf{b}^{(1)}\mathbf{b}^{(2)} \cdots \mathbf{b}^{(m)}]$$

requires

$$\frac{1}{3}n^3 + mn^2 - \frac{1}{3}n \quad \text{multiplications/divisions}$$

and

$$\frac{1}{3}n^3 + mn^2 - \frac{1}{2}n^2 - mn + \frac{1}{6}n \quad \text{additions/subtractions.}$$

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Maple has the command LUdecomp to compute a factorization of the form $A = PLU$ of the matrix A. If the matrix A has been created, the function call

```
>U:=LUdecomp(A,P='G', L='H');
```

returns the upper triangular matrix U as the value of the function and returns the lower triangular matrix L in H and the permutation matrix P in G.

EXERCISE SET 6.5

1. Solve the following linear systems:

 a. $$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

 b. $$\begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix}$$

2. Consider the following matrices. Find the permutation matrix P so that PA can be factored into the product LU, where L is lower triangular with 1's on its diagonal and U is upper triangular for these matrices.

 a. $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 4 & 0 \\ 0 & 1 & -1 \end{bmatrix}$

 b. $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$

 c. $A = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & 4 & 3 \\ 2 & -1 & 2 & 4 \\ 2 & -1 & 2 & 3 \end{bmatrix}$

 d. $A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 1 & -1 \\ 1 & 2 & -1 & 3 \\ 1 & 1 & 2 & 0 \end{bmatrix}$

3. Factor the following matrices into the LU decomposition using the LU Factorization Algorithm with $l_{ii} = 1$ for all i.

 a. $\begin{bmatrix} 2 & -1 & 1 \\ 3 & 3 & 9 \\ 3 & 3 & 5 \end{bmatrix}$

 b. $\begin{bmatrix} 1.012 & -2.132 & 3.104 \\ -2.132 & 4.906 & -7.013 \\ 3.104 & -7.013 & 0.014 \end{bmatrix}$

 c. $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 1.5 & 0 & 0 \\ 0 & -3 & 0.5 & 0 \\ 2 & -2 & 1 & 1 \end{bmatrix}$

 d. $\begin{bmatrix} 2.1756 & 4.0231 & -2.1732 & 5.1967 \\ -4.0231 & 6.0000 & 0 & 1.1973 \\ -1.0000 & -5.2107 & 1.1111 & 0 \\ 6.0235 & 7.0000 & 0 & -4.1561 \end{bmatrix}$

4. Modify the LU Factorization Algorithm so that it can be used to solve a linear system, and then solve the following linear systems.

Step 5 Set $k = k + 1$.

Step 6 For $i = 1, \ldots, n$ set $XO_i = x_i$.

Step 7 OUTPUT ('Maximum number of iterations exceeded');
(*The procedure was successful.*)
STOP. ■

EXERCISE SET 7.3

1. Find the first two iterations of the Jacobi method for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a. $$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b. $$\begin{aligned} 10x_1 - x_2 &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ - 2x_2 + 10x_3 &= 6. \end{aligned}$$

c. $$\begin{aligned} 10x_1 + 5x_2 &= 6, \\ 5x_1 + 10x_2 - 4x_3 &= 25, \\ - 4x_2 + 8x_3 - x_4 &= -11, \\ - x_3 + 5x_4 &= -11. \end{aligned}$$

d. $$\begin{aligned} 4x_1 + x_2 - x_3 + x_4 &= -2, \\ x_1 + 4x_2 - x_3 - x_4 &= -1, \\ -x_1 - x_2 + 5x_3 + x_4 &= 0, \\ x_1 - x_2 + x_3 + 3x_4 &= 1. \end{aligned}$$

e. $$\begin{aligned} 4x_1 + x_2 + x_3 + x_5 &= 6, \\ -x_1 - 3x_2 + x_3 + x_4 &= 6, \\ 2x_1 + x_2 + 5x_3 - x_4 - x_5 &= 6, \\ -x_1 - x_2 - x_3 + 4x_4 &= 6, \\ 2x_2 - x_3 + x_4 + 4x_5 &= 6. \end{aligned}$$

f. $$\begin{aligned} 4x_1 - x_2 - x_4 &= 0, \\ -x_1 + 4x_2 - x_3 - x_5 &= 5, \\ - x_2 + 4x_3 - x_6 &= 0, \\ -x_1 + 4x_4 - x_5 &= 6, \\ - x_2 - x_4 + 4x_5 - x_6 &= -2, \\ - x_3 - x_5 + 4x_6 &= 6. \end{aligned}$$

2. Repeat Exercise 1 using the Gauss-Seidel method. (b, c)

3. Use the Jacobi method to solve the linear systems in Exercise 1, with $TOL = 10^{-3}$ in the l_∞ norm.

4. Repeat Exercise 3 using the Gauss-Seidel Algorithm.

5. Find the first two iterations of the SOR method with $\omega = 1.1$ for the following linear systems, using $\mathbf{x}^{(0)} = \mathbf{0}$:

a. $$\begin{aligned} 3x_1 - x_2 + x_3 &= 1, \\ 3x_1 + 6x_2 + 2x_3 &= 0, \\ 3x_1 + 3x_2 + 7x_3 &= 4. \end{aligned}$$

b. $$\begin{aligned} 10x_1 - x_2 &= 9, \\ -x_1 + 10x_2 - 2x_3 &= 7, \\ - 2x_2 + 10x_3 &= 6. \end{aligned}$$

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Define three equations using the Maple commands

```
>eq1:=2*x1-3*x2+x3-4=0;
>eq2:=2*x1+x2-x3+4=0;
>eq3:=x1^2+x2^2+x3^2-4=0;
```

The third equation describes a sphere of radius 2 and center (0, 0, 0), so `x1`, `x2`, and `x3` are in $[-2, 2]$. The Maple commands to obtain the graph in this case are

```
>with(plots);
>implicitplot3d({eq1,eq2,eq3},x1=-2..2,x2=-2..2,x3=-2..2);
```

Various three-dimensional plotting options are available in Maple for isolating a solution to the nonlinear system. For example, we can rotate the graph to better view the sections of the surfaces. Then we can zoom into regions where the intersections lie and alter the display form of the axes for a more accurate view of the intersection's coordinates. For this problem, a reasonable initial approximation is $(x_1, x_2, x_3)^t = (-0.5, -1.5, 1.5)^t$.

EXERCISE SET 10.2

1. Use Newton's method with $\mathbf{x}^{(0)} = \mathbf{0}$ to compute $\mathbf{x}^{(2)}$ for each of the following nonlinear systems.

   a. $$4x_1^2 - 20x_1 + \frac{1}{4}x_2^2 + 8 = 0,$$
   $$\frac{1}{2}x_1x_2^2 + 2x_1 - 5x_2 + 8 = 0.$$

   b. $$\sin(4\pi x_1x_2) - 2x_2 - x_1 = 0,$$
   $$\left(\frac{4\pi - 1}{4\pi}\right)(e^{2x_1} - e) + 4ex_2^2 - 2ex_1 = 0.$$

   c. $$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0,$$
   $$4x_1^2 - 625x_2^2 + 2x_2 - 1 = 0,$$
   $$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0.$$

   d. $$x_1^2 + x_2 - 37 = 0,$$
   $$x_1 - x_2^2 - 5 = 0,$$
   $$x_1 + x_2 + x_3 - 3 = 0.$$

2. Use the graphing facilities of Maple to approximate solutions to the following nonlinear systems.

   a. $$x_1(1 - x_1) + 4x_2 = 12,$$
   $$(x_1 - 2)^2 + (2x_2 - 3)^2 = 25.$$

   b. $$5x_1^2 - x_2^2 = 0,$$
   $$x_2 - 0.25(\sin x_1 + \cos x_2) = 0.$$

   c. $$15x_1 + x_2^2 - 4x_3 = 13,$$
   $$x_1^2 + 10x_2 - x_3 = 11,$$
   $$x_2^3 - 25x_3 = -22.$$

   d. $$10x_1 - 2x_2^2 + x_2 - 2x_3 - 5 = 0,$$
   $$8x_2^2 + 4x_3^2 - 9 = 0,$$
   $$8x_2x_3 + 4 = 0.$$

3. Use Newton's method to find a solution to the following nonlinear systems in the given domain. Iterate until $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty < 10^{-6}$.

a. $$3x_1^2 - x_2^2 = 0,$$
$$3x_1x_2^2 - x_1^3 - 1 = 0.$$
Use $\mathbf{x}^{(0)} = (1, 1)^t$.

b. $$\ln(x_1^2 + x_2^2) - \sin(x_1x_2) = \ln 2 + \ln \pi,$$
$$e^{x_1 - x_2} + \cos(x_1x_2) = 0.$$
Use $\mathbf{x}^{(0)} = (2, 2)^t$.

c. $$x_1^3 + x_1^2x_2 - x_1x_3 + 6 = 0,$$
$$e^{x_1} + e^{x_2} - x_3 = 0,$$
$$x_2^2 - 2x_1x_3 = 4.$$
Use $\mathbf{x}^{(0)} = (-1, -2, 1)^t$.

d. $$6x_1 - 2\cos(x_2x_3) - 1 = 0,$$
$$9x_2 + \sqrt{x_1^2 + \sin x_3 + 1.06} + 0.9 = 0,$$
$$60x_3 + 3e^{-x_1x_2} + 10\pi - 3 = 0.$$
Use $\mathbf{x}^{(0)} = (0, 0, 0)^t$.

4. Use the answers obtained in Exercise 2 as initial approximations to Newton's method. Iterate until $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty < 10^{-6}$.

5. The nonlinear system

$$3x_1 - \cos(x_2x_3) - \frac{1}{2} = 0,$$
$$x_1^2 - 625x_2^2 - \frac{1}{4} = 0,$$
$$e^{-x_1x_2} + 20x_3 + \frac{10\pi - 3}{3} = 0$$

has a singular Jacobian matrix at the solution. Apply Newton's method with $\mathbf{x}^{(0)} = (1, 1 - 1)^t$. Note that convergence may be slow or may not occur within a reasonable number of iterations.

6. The nonlinear system

$$4x_1 - x_2 + x_3 = x_1x_4,$$
$$-x_1 + 3x_2 - 2x_3 = x_2x_4,$$
$$x_1 - 2x_2 + 3x_3 = x_3x_4,$$
$$x_1^2 + x_2^2 + x_3^2 = 1$$

has six solutions.

a. Show that if $(x_1, x_2, x_3, x_4)^t$ is a solution then $(-x_1, -x_2, -x_3, -x_4)^t$ is a solution.

b. Use Newton's method three times to approximate all solutions. Iterate until $\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|_\infty < 10^{-5}$.

7. Show that when $n = 1$, Newton's method given by Eq. (10.9) reduces to the familiar Newton's method given by Eq. (2.5).

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