1

Vector and tensor methods

1.1 THE FORMULAE OF VECTOR ANALYSIS SUMMARISED

In our previous book (O'Neill and Chorlton 1986), the main elements of vector field theory were developed together with some tensor analysis. Here a brief resumé is given of vector analysis and further work then follows on the tensor calculus, which is needed for the study of real fluid flows.

If $\overrightarrow{OA} \equiv \mathbf{a}$, $\overrightarrow{OB} \equiv \mathbf{b}$ and $A\hat{O}B = \theta$, then the **scalar product** of the two vectors **a** and **b** is defined to be $ab \cos \theta$ and is denoted by $\mathbf{a} \cdot \mathbf{b}$ so that

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a} = ab \cos \theta. \tag{1.1}$$

The reader will note that scalar product formation is *commutative*. With the two vectors localised at O, a plane AOB is formed and, at each point of the plane, two directions normal to it may be drawn and specified by the equal and opposite unit vectors $\pm\mathbf{n}$. If the direction $+\mathbf{n}$ is chosen to be in the sense of a *positive rotation* from **a** and **b** through θ, i.e. in the sense of a right-handed screw rotation from **a** to **b**, then we can form the **vector product** of **a** and **b**. This is usually donated by $\mathbf{a} \times \mathbf{b}$ but the alternative form $\mathbf{a} \wedge \mathbf{b}$ is sometimes used. It is defined to be

$$\mathbf{a} \times \mathbf{b} = ab \sin \theta \, \mathbf{n}. \tag{1.2}$$

From the definition we see that vector multiplication is *non-commutative* since

$$\mathbf{b} \times \mathbf{a} = (ba \sin \theta)(-\mathbf{n}) = -\mathbf{a} \times \mathbf{b}. \tag{1.3}$$

If now Ox, Oy, Oz form a tri-rectangular right-handed Cartesian coordinate frame and if $[a_1, a_2, a_3]$ denote the components of **a** in these axes and $[b_1, b_2, b_3]$ those of **b**, so that

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} = [a_1, a_2, a_3], \text{ etc.} \tag{1.4}$$

then the scalar product (1.1) in component form is

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3, \tag{1.5}$$

using $\mathbf{i}^2 = \mathbf{i}\cdot\mathbf{i}$, etc., and $\mathbf{i}\cdot\mathbf{j} = 0$, etc. Likewise the vector product (1.2) in component form becomes

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}, \tag{1.6}$$

which utilises such basic results as $\mathbf{i}\times\mathbf{i} = \mathbf{0}$, etc., $\mathbf{i}\times\mathbf{j} = -\mathbf{j}\times\mathbf{i} = \mathbf{k}$, etc. Here i, j, k are the unit vectors in $\overrightarrow{OX}$, $\overrightarrow{OY}$, $\overrightarrow{OZ}$.

If $\mathbf{c} = [c_1, c_2, c_3]$ is a third vector, then we can form the **scalar triple product** $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ of the vectors **a**, **b** and **c**. This is denoted by $[\mathbf{a}, \mathbf{b}, \mathbf{c}]$ and it is easy to show that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}. \tag{1.7}$$

Cyclic permutation of the vectors in a scalar triple product leaves it unchanged so that

$$[\mathbf{a}, \mathbf{b}, \mathbf{c}] = [\mathbf{b}, \mathbf{c}, \mathbf{a}] = [\mathbf{c}, \mathbf{a}, \mathbf{b}], \tag{1.8}$$

but their acyclic permutation results in a sign change:

$$[\mathbf{c}, \mathbf{b}, \mathbf{a}] = -[\mathbf{a}, \mathbf{b}, \mathbf{c}], \text{ etc.} \tag{1.9}$$

The vector product of **a** with the vector $\mathbf{b}\times\mathbf{c}$ is simply $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ and this is called the **vector triple product** of **a**, **b** and **c**. By using the component forms for **a**, etc., the rule for expansion of the vector triple product may be established in the form

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}. \tag{1.10}$$

At this stage, we proceed from the algebra to the calculus of vectors. If throughout a region of 3-space we have a single-valued differentiable scalar function $\phi(x, y, z)$ at each $P(x, y, z)$ of the region, then the **gradient** of ϕ is the vector function $\nabla\phi$ or

grad ϕ defined by

$$\nabla\phi = \text{grad } \phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}. \tag{1.11}$$

An alternative and equivalent form is

$$\nabla\phi = \frac{\partial\phi}{\partial n}\mathbf{n}, \tag{1.12}$$

where **n** is the unit vector along the normal at $P(x, y, z)$ to the level surface $\phi(x, y, z) = \text{constant}$ through P and directed from this level surface to the neighbouring one through $P'(x+\delta x, y+\delta y, z+\delta z)$. Letting $\overrightarrow{OP} \equiv \mathbf{r} = [x, y, z]$, and $\overrightarrow{OP}' \equiv \mathbf{r} + \delta\mathbf{r} = [x+\delta x, y+\delta y, z+\delta z]$, so that $\overrightarrow{PP}' \equiv \delta\mathbf{r} = [\delta x, \delta y, \delta z]$, then

$$\phi_{P'} - \phi = \delta\phi \approx \delta\mathbf{r}\cdot\nabla\phi$$

or, in the limiting form,

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\mathbf{r}\cdot\nabla\phi. \tag{1.13}$$

In the case of a vector function $\mathbf{F} = [F_1, F_2, F_3]$, where $F_n = F_n(x, y, z)$ (n = 1, 2, 3), one can obtain the **divergence** of **F**, denoted by div **F** or $\nabla\cdot\mathbf{F}$, a scalar function defined to be

$$\text{div } \mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}, \tag{1.14}$$

and also the **curl** of **F**, denoted by curl **F** or $\nabla\times\mathbf{F}$, a vector function defined to be

$$\text{curl } \mathbf{F} = \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}. \tag{1.15}$$

The **Laplacian** of ϕ is another scalar defined to be div grad ϕ or $\nabla^2\phi$, so that

$$\text{div grad } \phi = \nabla\cdot\nabla\phi = \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}. \tag{1.16}$$

A **harmonic function** ϕ satisfies **Laplace's equation** $\nabla^2\phi = 0$.

In the vector calculus a number of important vector identities arise. In addition to ϕ and $\mathbf{F}$ defined as above, we introduce a second differentiable function $\mathbf{G}$. Then the following identities hold:

$$\text{curl grad } \phi = \nabla \times \nabla\phi = \mathbf{0}, \tag{1.17}$$

$$\text{div curl } \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0, \tag{1.18}$$

$$\nabla \times (\nabla \times \mathbf{F}) = \nabla(\nabla \cdot \mathbf{F}) - \nabla^2\mathbf{F} \quad \text{or} \quad \text{curl curl } \mathbf{F} = \text{grad div } \mathbf{F} - \nabla^2\mathbf{F}, \tag{1.19}$$

$$\nabla \cdot (\mathbf{F} \times \mathbf{G}) = \mathbf{G} \cdot (\nabla \times \mathbf{F}) - \mathbf{F} \cdot (\nabla \times \mathbf{G}), \tag{1.20}$$

$$\nabla \times (\mathbf{F} \times \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} - (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{F}(\nabla \cdot \mathbf{G}) - \mathbf{G}(\nabla \cdot \mathbf{F}), \tag{1.21}$$

$$\nabla(\mathbf{F} \cdot \mathbf{G}) = (\mathbf{G} \cdot \nabla)\mathbf{F} + (\mathbf{F} \cdot \nabla)\mathbf{G} + \mathbf{G} \times (\nabla \times \mathbf{F}) + \mathbf{F} \times (\nabla \times \mathbf{G}), \tag{1.22}$$

$$(\mathbf{F} \cdot \nabla)\mathbf{F} = \nabla(\tfrac{1}{2}\mathbf{F}^2) - \mathbf{F} \times (\nabla \times \mathbf{F}), \tag{1.23}$$

$$\nabla^2\mathbf{F} = \mathbf{i}\,\nabla^2 F_1 + \mathbf{j}\,\nabla^2 F_2 + \mathbf{k}\,\nabla^2 F_3. \tag{1.24}$$

The last result is suitable only for Cartesian coordinates.

In the following, S is a closed surface containing a volume V and $\mathbf{n}$ is the unit vector to the surface element $\mathrm{d}S$ of S drawn outwards from V. The vector element of area is defined to be $\mathrm{d}\mathbf{S} = \mathrm{d}S\,\mathbf{n}$. The **Gauss divergence theorem** states that

$$\int_V \nabla \cdot \mathbf{F}\,\mathrm{d}V = \int_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_S \mathbf{n} \cdot \mathbf{F}\,\mathrm{d}S. \tag{1.25}$$

This leads to the alternative definition of div $\mathbf{F}$ at a point:

$$\text{div } \mathbf{F} = \lim_{V \to 0} \left(\frac{1}{V} \int_S \mathbf{n} \cdot \mathbf{F}\,\mathrm{d}S \right). \tag{1.26}$$

Immediate derivations stemming from the Gauss divergence theorem are

$$\int_V \nabla\phi\,\mathrm{d}V = \int_S \mathbf{n}\phi\,\mathrm{d}S = \int_S \phi\,\mathrm{d}\mathbf{S}, \tag{1.27}$$

$$\int_V \nabla \times \mathbf{F}\,\mathrm{d}V = \int_S \mathbf{n} \times \mathbf{F}\,\mathrm{d}S = \int_S (\mathrm{d}\mathbf{S} \times \mathbf{F}). \tag{1.28}$$

When $\mathscr{C}$ is a closed curve forming the rim of an open surface S, with vector arc element $\mathrm{d}\mathbf{r}$

$$\oint_{\mathscr{C}} \mathbf{F} \cdot \mathrm{d}\mathbf{r} = \int_S \mathbf{n} \cdot \text{curl } \mathbf{F}\,\mathrm{d}S. \tag{1.29}$$

This is **Stokes's theorem**.

1.2 GENERAL ORTHOGONAL CURVILINEAR COORDINATES

Suppose that at each $\mathrm{P}(x, y, z)$ of a region of 3-space there exist three uniform differentiable scalar functions $u_i(x, y, z)$ $(i = 1, 2, 3)$ having as level surfaces

$$u_i(x, y, z) = c_i \qquad (i = 1, 2, 3), \tag{1.3}$$

where each c_i is independent of x, y, z. Let us further suppose that these three surfaces are such that their curves of intersection through each P are mutually orthogonal. Then (u_1, u_2, u_3) forms a system of coordinates alternative to the rectangular Cartesian system (x, y, z): they are called the **general orthogonal curvilinear coordinates** of P.

In rectangular Cartesian coordinates (x, y, z) the vector arc element, denoted either by $\mathrm{d}\mathbf{r}$ or $\mathrm{d}\mathbf{s}$, is given by

$$\mathrm{d}\mathbf{r} = \mathrm{d}x\,\mathbf{i} + \mathrm{d}y\,\mathbf{j} + \mathrm{d}z\,\mathbf{k}. \tag{1.31}$$

The corresonding form in general orthogonal curvilinear coordinares (u_1, u_2, u_3) is

$$\mathrm{d}\mathbf{r} = h_1\,\mathrm{d}u_1\,\hat{\mathbf{a}}_1 + h_2\,\mathrm{d}u_2\,\hat{\mathbf{a}} + h_3\,\mathrm{d}u_3\,\hat{\mathbf{a}}_3, \tag{1.32}$$

involving the three scale factors $h_i(u_1, u_2, u_3)$ and the three unit vectors $\hat{\mathbf{a}}_i$ $(i = 1, 2, 3)$ such that, for $i, j = 1, 2, 3$,

$$\hat{\mathbf{a}}_i^{\,2} = 1; \qquad \hat{\mathbf{a}}_i \cdot \hat{\mathbf{a}}_j = 0 \text{ for } i \neq j. \tag{1.33}$$

Equations (1.33) express orthonormal relations between the unit vectors which are essentially tangential to the curves of intersection of the three surfaces (1.30)

For the special case of rectangular coordinates (x, y, z) orthogonality holds and we may take $u_1 = x$, $u_2 = y$, $u_3 = z$; $\hat{\mathbf{a}}_i = \mathbf{i}$, $\hat{\mathbf{a}}_2 = \mathrm{j}$, $\hat{\mathbf{a}}_3 = \mathbf{k}$. Comparison of the forms (1.34) and (1.35) shows that, for the Cartesian system, $h_1 = h_2 = h_3 = 1$.

The vector functions grad, div and curl may be expressed in terms of the general orthogonal coordinates u_i and the scale factors h_i which enter through the form (1.32) $(i = 1, 2, 3)$. In the following, $U = U(u_1, u_2, u_3)$ is an appropriate scalar function and $\mathbf{F}(u_1, u_2, u_3) = F_i(u_1, u_2, u_3)\hat{\mathbf{a}}_i$ is a suitable vector function, where $\hat{\mathbf{a}}_i = \hat{\mathbf{a}}_i(u_1, u_2, u_3)$ $(i = 1, 2, 3)$. All relevant derivatives for the formation of gradient, divergence, curl and Laplacians are supposed to exist so that

$$\nabla U = \frac{1}{h_1}\frac{\partial U}{\partial u_1}\hat{\mathbf{a}}_1 + \frac{1}{h_2}\frac{\partial U}{\partial u_2}\hat{\mathbf{a}}_2 + \frac{1}{h_3}\frac{\partial U}{\partial u_3}\hat{\mathbf{a}}_3, \tag{1.34}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(h_2 h_3 F_1) + \frac{\partial}{\partial u_2}(h_3 h_1 F_2) + \frac{\partial}{\partial u_3}(h_1 h_2 F_3)\right]. \tag{1.35}$$

$$\nabla \times \mathbf{F} = \frac{1}{h_1 h_2 h_3} \begin{vmatrix} h_1 \hat{\mathbf{a}}_1 & h_2 \hat{\mathbf{a}}_2 & h_3 \hat{\mathbf{a}}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 F_1 & h_2 F_2 & h_3 F_3 \end{vmatrix}, \tag{1.36}$$

$$\nabla^2 U = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial U}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial U}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial U}{\partial u_3} \right) \right]. \tag{1.37}$$

For the particular orthogonal systems of cylindrical and spherical polar coordinates, we now give the forms corresponding to (1.34)–(1.37).

1.2.1 Cylindrical polar coordinates (R, ϕ, z)

We take $u = R$, $u_2 = \phi$, $u_3 = z$ and the line element

$$\mathrm{d}\mathbf{r} = \mathrm{d}R\, \hat{\mathbf{R}} + R\, \mathrm{d}\phi\, \boldsymbol{\phi} + \mathrm{d}z\, \mathbf{k}. \tag{1.38}$$

Hence the scale factors for this system are

$$h_R = 1, \qquad h_\phi = R, \qquad h_z = 1. \tag{1.39}$$

Hence, for $U = U(R, \phi, z)$, $\mathbf{F} = F_R \hat{\mathbf{R}} + F_\phi \hat{\boldsymbol{\phi}} + F_z \mathbf{k}$, where $F_R = F_R(R, \phi, z)$, etc., the formulae (1.34)–(1.37) inclusive give

$$\nabla U = \frac{\partial U}{\partial R} \hat{\mathbf{R}} + \frac{1}{R} \frac{\partial U}{\partial \phi} \hat{\boldsymbol{\phi}} + \frac{\partial U}{\partial z} \mathbf{k}, \tag{1.40}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{R} \left(\frac{\partial}{\partial R} (R F_R) + \frac{\partial F_\phi}{\partial \phi} + R \frac{\partial F_z}{\partial z} \right), \tag{1.41}$$

$$\nabla \times \mathbf{F} = \frac{1}{R} \begin{vmatrix} \hat{\mathbf{R}} & R\hat{\boldsymbol{\phi}} & \mathbf{k} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ F_R & RF_\phi & F_z \end{vmatrix}, \tag{1.42}$$

$$\nabla^2 U = \frac{1}{R} \left[\frac{\partial}{\partial R} \left(R \frac{\partial U}{\partial R} \right) + \frac{1}{R} \frac{\partial^2 U}{\partial \phi^2} + R \frac{\partial^2 U}{\partial z^2} \right]. \tag{1.43}$$

1.2.2 Spherical polar coordinates (r, θ, ϕ)

We take $u_1 = r$, $u_2 = \theta$, $u_3 = \phi$ and the line element

$$\mathrm{d}\mathbf{r} = \mathrm{d}r\, \hat{\mathbf{r}} + r\, \mathrm{d}\theta\, \hat{\boldsymbol{\theta}} + r \sin\theta\, \mathrm{d}\phi\, \hat{\boldsymbol{\phi}}. \tag{1.44}$$

The scale factors for the system are

$$h_r = 1, \qquad h_\theta = r, \qquad h_\phi = r \sin\theta. \tag{1.45}$$

Taking $U = U(r, \theta, \phi)$, $\mathbf{F} = F_r \hat{\mathbf{r}} + F_\theta \hat{\boldsymbol{\theta}} + F_\phi \hat{\boldsymbol{\phi}}$, where $F_r = F_r(r, \theta, \phi)$, etc., we find that

$$\nabla U = \frac{\partial U}{\partial r} \hat{\mathbf{r}} + \frac{1}{r} \frac{\partial U}{\partial \theta} \hat{\boldsymbol{\theta}} + \frac{1}{r \sin\theta} \frac{\partial U}{\partial \phi} \hat{\boldsymbol{\phi}}, \tag{1.46}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{\partial (r^2 F_r)}{\partial r} + r \frac{\partial (\sin\theta\, F_\theta)}{\partial \theta} + \frac{\partial (r F_\phi)}{\partial \phi} \right], \tag{1.47}$$

$$\nabla \times \mathbf{F} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\mathbf{r}} & r\hat{\boldsymbol{\theta}} & r \sin\theta\, \hat{\boldsymbol{\phi}} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ F_r & rF_\theta & r \sin\theta\, F_\phi \end{vmatrix}, \tag{1.48}$$

$$\nabla^2 U = \frac{1}{r^2 \sin\theta} \left[\sin\theta \frac{\partial}{\partial r} \left(r^2 \frac{\partial U}{\partial r} \right) + \frac{\partial}{\partial \theta} \left(\sin\theta \frac{\partial U}{\partial \theta} \right) + \operatorname{cosec}\theta \frac{\partial^2 U}{\partial \phi^2} \right]. \tag{1.49}$$

1.3 CONTRACTED NOTATION AND THE SUMMATION CONVENTION

In the abridged notation, we write

$$\mathbf{a} = a_1 \hat{\mathbf{e}}_1 + a_2 \hat{\mathbf{e}}_2 + a_3 \hat{\mathbf{e}}_3 = a_i \hat{\mathbf{e}}_i, \tag{1.50}$$

$$\mathbf{b} = b_1 \hat{\mathbf{e}}_1 + b_2 \hat{\mathbf{e}}_2 + b_3 \hat{\mathbf{e}}_3 = b_i \hat{\mathbf{e}}_i, \tag{1.51}$$

the vectors $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$ and $\hat{\mathbf{e}}_3$ now denoting the Cartesian unit vectors formerly written as **i**, **j** and **k**, respectively, and being in the positive directions of the x, y and z axes. It will also be convenient to denote the coordinates (x, y, z) by (x_1, x_2, x_3), respectively. For a right-handed frame, $\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2 = \hat{\mathbf{e}}_3$, etc. The scalar product $\mathbf{a} \cdot \mathbf{b}$ is simply

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3 = a_i b_i. \tag{1.52}$$

Note that, in (1.50)–(1.52), we have introduced the **summation convention** for repeated subscripts in each term on the far right-hand side.

To illustrate further the summation convention let us consider the meaning of $a_{ij} x_j$, where both i and j can range through the values 1, 2, 3. Firstly, j is a **repeated subscript** which means that summation takes place with respect to it from $j = 1$ to $j = 3$, thereby generating $a_{i1} x_1 + a_{i2} x_2 + a_{i3} x_3$. When i is allowed to assume each of the values 1, 2, 3 separately, the three sums

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3,$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3,$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3,$$

are generated. Hence the terse form $a_{ij}x_j$ means these three sums. The repeated subscript is a dummy and generates summation. It must not be repeated more than once. We note that $a_{ij}x_j = a_{ik}x_k$, since j and k are both dummy subscripts. The subscript i is free and must not be replaced by any other subscript.

The **Kronecker delta** is defined to be

$$\delta_{ij} = 1 \text{ when } i = j \text{ (no summation)}, \qquad = 0 \text{ when } i \neq j. \tag{1.53}$$

Then $\delta_{23} = 0$, $\delta_{22} = 1$,etc., but

$$\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3.$$

For the form $\delta_{ij}x_j$, we have

$$\delta_{ij}x_j = \delta_{i1}x_1 + \delta_{i2}x_2 + \delta_{i3}x_3.$$

When $i = 1$, the right-hand side becomes $\delta_{11}x_1 + \delta_{12}x_2 + \delta_{12}x_3 = x_1$. Similarly, when $i = 2$, $\delta_{2j}x_j = x_2$ and, when $i = 3$, $\delta_{3j}x_j = x_3$. Thus we have the important result that

$$\delta_{ij}x_j = x_i \tag{1.54}$$

which shows that the action of δ_{ij} on x_j is to substitute the free i for the dummy j.

In our previous book (O'Neill and Chorlton 1986, pp. 72–73), it is shown that, if the three coordinate axes Ox_i $(i = 1, 2, 3)$ form a right-handed orthogonal coordinate frame and if this frame undergoes rotation about O to new positions specified by Ox_i' $(i = 1, 2, 3)$ such that the newly positioned axis Ox_i' has the direction cosines $[l_{i1}, l_{i2}, l_{i3}]$ with respect to the former frame, then Ox_i has direction cosines $[l_{1i}, l_{2i}, l_{3i}]$ with respect to the primed axes. The matrix $[l_{ij}]$ is called the **transformation matrix** of the rotation. The **orthonormal properties** of the l values are established in the forms

$$l_{ir}l_{is} = \delta_{rs} = l_{ri}l_{si}. \tag{1.55}$$

In (1.55), r and s are both free subscripts and i is a dummy subscript. Full details of these developments appear in our previous book, and we now make use of them in section 1.4.

1.4 CARTESIAN TENSOR OF ORDER n

Let $u_{ijk\ldots}$ be a quantity involving n subscripts $i, j, k, \ldots$ and suppose that, under a right-handed orthogonal rotation of coordinate axes, it changes to the form $u'_{pqr\ldots}$ involving n new subscripts $p, q, r, \ldots$. Under this type of transformation we suppose that the orthonormal relations described in (1.55) hold and that

$$u'_{pqr\ldots} = l_{pi}l_{qj}l_{rk}\ldots u_{ijk\ldots} \tag{1.56}$$

involving n of the l values on the right-hand side. Then $u_{ijk\ldots}$ is called a **tensor of order (or rank)** n. Equation (1.56), coupled with (1.55), expresses the **law of transformation** of such a tensor.

From (1.55) and (1.56), we prove that

$$u_{pqr\ldots} = l_{ip}l_{jq}l_{kr}\ldots u'_{ijk\ldots}. \tag{1.57}$$

Relabelling the subscripts in (1.56) gives

$$u'_{ijk\ldots} = l_{i\alpha}l_{j\beta}l_{k\gamma}\ldots u_{\alpha\beta\gamma\ldots}$$

and so the right-hand side of (1.57) becomes

$$\begin{aligned}(l_{ip}l_{jq}l_{kr}\ldots)(l_{i\alpha}l_{j\beta}l_{k\gamma}\ldots u_{\alpha\beta\gamma\ldots}) &= (l_{ip}l_{i\alpha})(l_{jq}l_{j\beta})(l_{kr}l_{k\gamma})\cdots u_{\alpha\beta\gamma\ldots} \\ &= \delta_{p\alpha}\delta_{q\beta}\delta_{r\gamma}\cdots u_{\alpha\beta\gamma\ldots} \\ &= u_{pqr\ldots},\end{aligned}$$

on using the substitution properties of the Kronecker deltas. Conversely, it is easy to show by the same devices that starting from (1.55) and (1.57), equation (1.56) follows.

If, starting from the nth-order tensor $u_{ijk\ldots}$, we write $j = i$, then

$$u_{iik\ldots} = u_{11k\ldots} + u_{22k\ldots} + u_{33k\ldots},$$

since i is repeated once only so that summation takes place with respect to this dummy subscript, the remaining $n - 2$ subscripts being free. We prove, starting from (1.55) and (1.56), that $u_{iik\ldots}$ is a Cartesian tensor of order $n - 2$. This requires showing that it obeys the law of transformation of a tensor of order $n - 2$. To this end, we consider the expression

$$l_{rk}l_{sm}\ldots u_{iikm\ldots}$$

involving $n - 2$ of the l values and $n - 2$ free subscripts $r, s, \ldots$. Since

$$\begin{aligned}u_{iikm\ldots} &= \delta_{ij}u_{ijkm\ldots} \\ &= l_{pi}l_{pj}u_{ijkm\ldots} \\ &= \delta_{pq}l_{pi}l_{qj}u_{ijkm\ldots},\end{aligned}$$

we have

$$\begin{aligned}(l_{rk}l_{sm\dots})u_{iikm\dots} &= (l_{rk}l_{sm\dots})(\delta_{pq}l_{pi}l_{qj}u_{ijkm\dots})\\ &= \delta_{pq}(l_{pi}l_{qj}l_{rk}l_{sm\dots})u_{ijkm\dots}\\ &= \delta_{pq}u'_{pqrs\dots}\\ &= u'_{pprs\dots}.\end{aligned}$$

Thus

$$u'_{pprs\dots} = l_{rk}l_{sm\dots}u_{iikm\dots}. \tag{1.58}$$

This equation, together with (1.55), expresses the law of transformation of a Cartesian tensor of order $n-2$, as required.

The process of obtaining from the given nth-order tensor $u_{ijk\dots}$ a tensor of order $n-2$ by taking $j=i$ is equivalent to forming $\delta_{ij}u_{ijk\dots}$ and is known as **contraction**. Such a process applied to a second-order tensor u_{ij} gives $\delta_{ij}u_{ij} = u_{ii} = u_{11} + u_{22} + u_{33}$, which is a tensor of order zero, i.e. a scalar.

1.5 THE ALTERNATING SYMBOL ε_{ijk}

Starting from the set of numbers (1, 2, 3), we can permute them cyclically to give the three **even permutations**

$$(1, 2, 3);\ (2, 3, 1);\ (3, 1, 2).$$

If, however, starting from (1, 2, 3), we exchange any two elements leaving the other *in situ*, then we generate a group of acyclic or **odd permutations,** i.e.

$$(2, 1, 3);\ (1, 3, 2);\ (3, 2, 1).$$

Any other grouping of the three numbers (1, 2, 3), such as (2, 2, 3), is not a permutation of them at all.

The **alternating symbol** ε_{ijk} is defined to be

$$\begin{aligned}\varepsilon_{ijk} &= +1 \text{ when } (i, j, k) \text{ is an even permutation of } (1, 2, 3),\\ &= -1 \text{ when } (i, j, k) \text{ is an odd permutation of } (1, 2, 3),\\ &= \ \ 0 \text{ when } (i, j, k) \text{ is not a permutation of } (1, 2, 3).\end{aligned}$$

Thus $\varepsilon_{312} = 1$, $\varepsilon_{132} = -1$, and $\varepsilon_{121} = 0$.

The third-order determinant

$$\Delta = \det a_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} \tag{1.59}$$

has an expansion of the form $\varepsilon_{ijk}a_{1i}a_{2j}a_{3k}$. Using the summation convention,

$$\begin{aligned}\varepsilon_{ijk}a_{1i}a_{2j}a_{3k} &= \varepsilon_{123}a_{11}a_{22}a_{33} + \varepsilon_{132}a_{11}a_{23}a_{32}\\ &\quad + \varepsilon_{213}a_{12}a_{21}a_{33} + \varepsilon_{231}a_{12}a_{23}a_{31}\\ &\quad + \varepsilon_{312}a_{13}a_{21}a_{32} + \varepsilon_{321}a_{13}a_{22}a_{31},\end{aligned}$$

where zero terms involving ε_{112}, etc., have been omitted. The last expression simplifies to

$$a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}),$$

which is the expansion of Δ across its top row. In fact the form $\varepsilon_{ijk}a_{1i}a_{2j}a_{3k}$ can be used as a definition of Δ as indeed it often is in more sophisticated treatments of determinants.

Starting with the determinantal form (1.59), let us interchange rows $(1, r)$; $(2, s)$; $(3, t)$ to generate the new determinant Δ' defined to be

$$\Delta' = \begin{vmatrix} a_{r1} & a_{r2} & a_{r3}\\ a_{s1} & a_{s2} & a_{s3}\\ a_{t1} & a_{t2} & a_{t3}\end{vmatrix}. \tag{1.60}$$

Since $\Delta = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k}$, $\Delta' = \varepsilon_{ijk}a_{ri}a_{sj}a_{tk}$, a mere replacement of 1, 2, 3 by r, s, t, respectively. How is Δ' related to Δ? From elementary determinantal theory an even number of row interchanges of a determinant leaves the determinant unchanged. Hence, if (r, s, t) is an even permutation of (1, 2, 3), then $\Delta' = \Delta$. If it is an odd permutation, then $\Delta' = -\Delta$. If two rows are equal in Δ', then $\Delta' = 0$ and moreover (r, s, t) is not then a permutation of (1, 2, 3) at all. Combining the three cases gives

$$\Delta' = \varepsilon_{rst}\Delta. \tag{1.61}$$

Hence,

$$\varepsilon_{rst}\Delta = \varepsilon_{ijk}a_{ri}a_{sj}a_{tk} = \varepsilon_{ijk}a_{ir}a_{js}a_{kt}. \tag{1.62}$$

The last form in (1.62) follows by recalling that Δ is unchanged when the elements of its matrix are transposed so that

$$\Delta = \varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \varepsilon_{ijk}a_{i1}a_{j2}a_{k3}. \tag{1.63}$$

It is of some importance to establish connections between the δ and ε symbols. We prove the following results:

$$\varepsilon_{ijk}\varepsilon_{rsk} = \delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}, \tag{1.64}$$

$$\varepsilon_{ijk}\varepsilon_{rjk} = 2\delta_{ir}, \tag{1.65}$$

$$\varepsilon_{ijk}\varepsilon_{ijk} = 6. \tag{1.66}$$

The forms cited suggest that we first evaluate ε_{rst} and then $\varepsilon_{ijk}\varepsilon_{rst}$. Since

$$\varepsilon_{ijk}a_{1i}a_{2j}a_{3k} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix},$$

$$\varepsilon_{ijk}\delta_{1i}\delta_{2j}\delta_{3k} = \begin{vmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{vmatrix} = 1.$$

Interchange of rows $(1, r)$; $(2, s)$; $(3, k)$ in the last form gives

$$\varepsilon_{ijk}\delta_{ri}\delta_{sj}\delta_{tk} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix},$$

i.e.

$$\varepsilon_{rst} = \begin{vmatrix} \delta_{r1} & \delta_{r2} & \delta_{r3} \\ \delta_{s1} & \delta_{s2} & \delta_{s3} \\ \delta_{t1} & \delta_{t2} & \delta_{t3} \end{vmatrix}.$$

The operation of ε_{ijk} on ε_{rst} will interchange the columns $(1, i)$; $(2, j)$; $(3, k)$ in the last determinant to give

$$\varepsilon_{ijk}\varepsilon_{rst} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ti} & \delta_{tj} & \delta_{tk} \end{vmatrix}.$$

In the last form, take $t = k$ to give

$$\varepsilon_{ijk}\varepsilon_{rsk} = \begin{vmatrix} \delta_{ri} & \delta_{rj} & \delta_{rk} \\ \delta_{si} & \delta_{sj} & \delta_{sk} \\ \delta_{ki} & \delta_{kj} & 3 \end{vmatrix}$$

$$= \delta_{ri}(3\delta_{sj} - \delta_{sk}\delta_{kj}) - \delta_{rj}(3\delta_{si} - \delta_{sk}\delta_{ki}) + \delta_{rk}(\delta_{si}\delta_{kj} - \delta_{sj}\delta_{ki})$$
$$= 3\delta_{ri}\delta_{sj} - \delta_{ri}\delta_{sj} - 3\delta_{rj}\delta_{si} + \delta_{rj}\delta_{si} + \delta_{si}\delta_{rj} - \delta_{sj}\delta_{ri}$$
$$= \delta_{ri}\delta_{sj} - \delta_{rj}\delta_{si}.$$

When we recall that $\delta_{ij} = \delta_{ji}$, we see that the last form is accordingly the right-hand side of (1.64).

Next, take $j = s$ in (1.64). Then

$$\varepsilon_{isk}\varepsilon_{rsk} = \delta_{ir}(3) - \delta_{is}\delta_{sr} = 3\delta_{ir} - \delta_{ir} = 2\delta_{ir},$$

which establishes (1.65). Finally take $i = r$ in (1.65) to give (1.66).

1.6 PROPER TENSORS AND PSEUDO-TENSORS

Reverting to section 1.3, we suppose that $[l_{ij}]$ is the matrix of a transformation which is orthogonal but not necessarily right handed, so that the original right-handed orthogonal frame OX_i' may be transformed into another orthogonal frame Ox_i' which may be either right or left handed. Then the determinant of the transform is

$$\Delta = \det l_{ij} = \pm 1, \tag{1.67}$$

and the orthonormal relations

$$l_{ri}l_{si} = \delta_{rs} = l_{ir}l_{is} \tag{1.68}$$

still prevail. Thus, since $\varepsilon_{ijk}l_{1i}l_{2j}l_{3k} = \Delta$,

$$l_{ri}l_{sj}l_{tk}\varepsilon_{ijk} = \varepsilon_{rst}\Delta, \tag{1.69}$$

since the left-hand side of (1.69) is the determinant derived from Δ by interchanging the pairs of rows $(1, r)$; $(2, s)$; $(3, t)$. From (1.67) and (1.69), we may also write

$$\varepsilon_{rst} = \Delta l_{ri}l_{sj}l_{tk}\varepsilon_{ijk}. \tag{1.70}$$

For a specifically right-handed rotation, $\Delta = +1$ and (1.70) in conjunction with (1.68) expresses the law of transformation of a third-order tensor. However, a change from a right- to a left-handed frame makes $\Delta = -1$. Thus the nature of the transformation of ε_{ijk} into ε_{rst} depends not only on the perpetuated orthogonality of the axes but also on whether they stay right handed or change to left handed. Such a quantity ε_{ijk} is thus termed a **pseudo-tensor** of the third order. It is also sometimes called an **axial tensor** because of its dependence on the nature of the axes—right- or left-handed. The term **tensor density** is also used. A tensor for which the law of transformation holds unchanged for both right- and left-handed orthogonal transforms is called a

proper tensor. An example of a second-order proper tensor is afforded by δ_{ij} since $\delta_{rs} = l_{ri}l_{sj}\delta_{ij}$. We note that ε_{ijk} is a symmetric pseudo-tensor of the third order, since the interchange of any two subscripts leaving the third *in situ* changes its sign.

Let us extend the previous remarks about pseudo-tensors. If a_{ij} is a proper tensor of the second order, then its law of transformation is

$$a_{rs}' = l_{ri}l_{sj}s_{ij},$$

subject to the orthonormal relations (1.68). If, however, its law of transformation were

$$a_{rs}^* = \Delta l_{ri}l_{sj}a_{ij},$$

where $\Delta = \det l_{ij} = \pm 1$, then, as the transform is now axially dependent in the sense of left- and right-handedness, a_{ij} is a pseudo-tensor of the second order. To generalise, we may say that the law of transformation of a pseudo-tensor of the nth order of the form $a_{ijk\ldots}$ involving subscripts $i, j, k, \ldots$ to $a_{rst\ldots}^*$ is

$$a_{rst\ldots}^* = \Delta l_{ri}l_{sj}l_{tk}\cdots a_{ijk\ldots}, \tag{1.71}$$

subject to (1.68), where $\Delta = \det l_{ij} = \pm 1$. It is easy to establish the following results.

(1) The sum or difference of two pseudo-tensors of the same order is another pseudo-tensor of that order.
(2) The product of a pseudo-tensor with a proper tensor is another pseudo-tensor.
(3) The product of two pseudo-tensors is a proper tensor.
(4) A contracted pseudo-tensor of order n ($\geqslant 2$) is another pseudo-tensor of the order $n-2$.

Thus, to illustrate result (3), if a_{ijk} and b_{pq} are pseudo-tensors of orders 3 and 2 respectively, then

$$\begin{aligned} a_{rst}^* b_{mn}^* &= (\Delta l_{ri}l_{sj}l_{tk}a_{ijk})(\Delta l_{m\alpha}l_{n\beta}b_{\alpha\beta}) \\ &= l_{ri}l_{sj}l_{tk}l_{m\alpha}l_{n\beta}(a_{ijk}b_{\alpha\beta}). \end{aligned}$$

This shows that $a_{ijk}b_{\alpha\beta}$ is a proper tensor of order 5.

As a further illustration, and one which is important in continuum mechanics, suppose that a_{ij} is a skew-symmetric second-order proper tensor and let us consider

$$a_i^* = \tfrac{1}{2}\varepsilon_{ijk}a_{jk}. \tag{1.72}$$

By virtue of result (2), $\varepsilon_{ijk}a_{mn}$ is a pseudo-tensor of order 5. Taking $m = j$, $n = k$ contracts it to a pseudo-tensor of order unity or pseudo-vector, which is therefore what (1.72) is. The factor of $\frac{1}{2}$ is a mere convenience. Writing out the components

of (1.72) fully gives

$$\begin{aligned} a_1^* &= \tfrac{1}{2}\varepsilon_{123}a_{23} + \tfrac{1}{2}\varepsilon_{132}a_{32} = a_{23}, \\ a_2^* &= \tfrac{1}{2}\varepsilon_{231}a_{31} + \tfrac{1}{2}\varepsilon_{213}a_{13} = a_{31}, \\ a_3^* &= \tfrac{1}{2}\varepsilon_{312}a_{12} + \tfrac{1}{2}\varepsilon_{321}a_{21} = a_{12}. \end{aligned} \tag{1.73}$$

Thus we can make the following statement.

The three components of a skew-symmetric second-order proper tensor are the components of a pseudo-vector.

In particular, the three components of the vector product $\mathbf{b} \times \mathbf{c}$, where $\mathbf{b}$ has components b_i and $\mathbf{c}$ has components c_i, are

$$[(b_2c_3 - b_3c_2), (b_3c_1 - b_1c_3), (b_1c_2 - b_2c_1)].$$

These are the components a_{ij} of a skew-symmetric proper tensor of the second order, where

$$a_{ij} = b_ic_j - b_jc_i = -a_{ji}.$$

Thus we may represent $\mathbf{b} \times \mathbf{c}$ by the 3×3 skew-symmetric matrix

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ a_{21} & 0 & a_{23} \\ a_{31} & a_{32} & 0 \end{bmatrix}.$$

This means that $\mathbf{b} \times \mathbf{c}$ is not a proper vector although, of course, if we adhere to right-handed frames, then no difference emerges between it and a proper vector. To illustrate the difference further, if the transformation matrix $\Delta = [-\delta_{ij}]$, then, for the vector $\mathbf{b}$,

$$b_r' = l_{ri}b_i = -\delta_{ri}b_i = -b_r,$$

i.e. in the new frame the components of $\mathbf{b}$ are $[-b_1, -b_2, -b_3]$. Similarly, those of $\mathbf{c}$ are $[-c_1, -c_2, -c_3]$. Thus

$$b_i'c_j' - b_j'c_i' = b_ic_j - b_jc_i,$$

i.e. the components of $\mathbf{b} \times \mathbf{c}$ remain unaltered under this transformation, whereas those of $\mathbf{b}$ and $\mathbf{c}$ undergo changes of sign. It will be recalled that, in defining $\mathbf{b} \times \mathbf{c}$, we had to stipulate a right-handed rotation from $\mathbf{b}$ to $\mathbf{c}$. This is not necessary for a proper vector. Then we have the following:

The vector product $\mathbf{b} \times \mathbf{c}$ may be regarded either as a proper skew-symmetric tensor with components $a_{ij} = b_i c_j - b_j c_i$ or as a pseudo-vector with components $a_i^* = \frac{1}{2}\varepsilon_{ijk} b_j c_k$.

1.7 ROTATION ABOUT A FIXED LINE

In Fig. 1.1, Ox_i are the axes of a right-handed tri-rectangular frame and a fixed line through O is specified in direction by the unit vector $\mathbf{n} = n_i \hat{\mathbf{e}}_i$ $(i = 1, 2, 3)$. A rigid

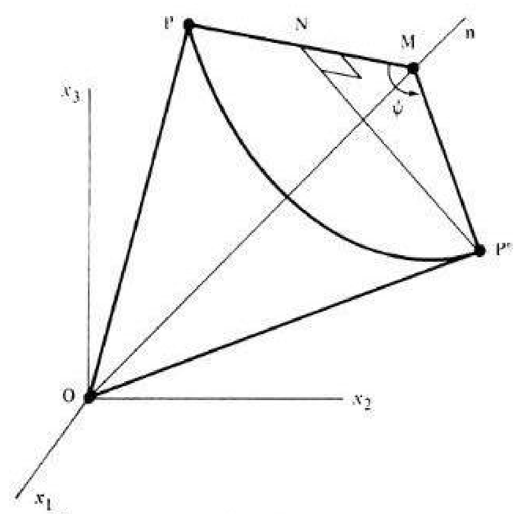

Fig. 1.1

body type of rotation takes place about the fixed line through an angle ψ so that $P(x_i)$ in the body travels to $P'(x_i')$. Then

$$\overrightarrow{OP'} = \overrightarrow{OP} + \overrightarrow{PN'} + \overrightarrow{NP'} \tag{1.74}$$

where PM and P'M are both perpendicular to $\mathbf{n}$ so that $P\hat{M}P' = \psi$, PM = P'M and P'N is perpendicular to PM. We denote $\overrightarrow{OP}$ by $\mathbf{r}$ and $\overrightarrow{OP'}$ by $\mathbf{r}'$ and we first find $\mathbf{r}'$ in terms of $\mathbf{r}$, $\mathbf{n}$ and $\mathbf{n} \times \mathbf{r}$. We have

$$\overrightarrow{PM} = \overrightarrow{OM} - \overrightarrow{OP} \equiv (\mathbf{r}\cdot\mathbf{n})\mathbf{n} - \mathbf{r} = \mathbf{v} \text{ (say)}$$

and so

$$\overrightarrow{PN} = \frac{PN}{PM}\overrightarrow{PM} \equiv (1 - \cos\psi)[(\mathbf{r}\cdot\mathbf{n})\mathbf{n} - \mathbf{r}]. \tag{1.75}$$

Next, NP' is at right angles to both $\mathbf{v}$ and $\mathbf{n}$ and $\mathbf{v} \times \mathbf{n}$ is a vector in the direction $\overrightarrow{NP'}$, i.e. $-\mathbf{r} \times \mathbf{n}$ is such a vector. Since $|\mathbf{v} \times \mathbf{n}| = PM$, the unit vector in $\overrightarrow{NP'}$ is $(\mathbf{n} \times \mathbf{r})/PM$ and so

$$\overrightarrow{NP'} = \frac{NP'}{PM}(\mathbf{n} \times \mathbf{r}) \equiv \sin\psi\,(\mathbf{n} \times \mathbf{r}). \tag{1.76}$$

Substituting (1.75) and (1.76) into (1.74) gives

$$\mathbf{r}' = \mathbf{r}\cos\psi + (1 - \cos\psi)(\mathbf{r}\cdot\mathbf{n})\mathbf{n} + (\sin\psi)\mathbf{n} \times \mathbf{r}. \tag{1.77}$$

Equating the ith components in (1.77) gives

$$x_i' = x_i \cos\psi + (1 - \cos\psi) n_i n_j x_j + \sin\psi\,\varepsilon_{ijk} n_j x_k.$$

Since $\varepsilon_{ijk} n_j x_k = \varepsilon_{ikj} n_k x_j = -\varepsilon_{ijk} n_k x_j$, the last form may be written

$$x_i' = a_{ij} x_j, \tag{1.78}$$

where

$$a_{ij} = \cos\psi\,\delta_{ij} + (1 - \cos\psi) n_i n_j - \sin\psi\,\varepsilon_{ijk} n_k \tag{1.79}$$

When an infinitesimal rotation is carried out, ψ is small and, to a first order of approximation, $\cos\psi \approx 1$ and $\sin\psi \approx \psi$. Thus transformation now approximates to

$$x_i' = x_i + s_{ik} x_k, \qquad s_{ik} = \psi \varepsilon_{ijk} n_j. \tag{1.80}$$

Clearly, s_{ik} is a skew-symmetric pseudo-tensor of the second order.

The solution to (1.80) for the determination of the axis of rotation is uniquely

$$n_1 = \frac{-s_{23}}{\psi}, \qquad n_2 = \frac{-s_{31}}{\psi}, \qquad n_3 = \frac{-s_{12}}{\psi}.$$

The case of infinitesimal rotations is of great importance in linear elasticity theory.

1.8 ISOTROPIC TENSORS

An **isotropic tensor** is one which transforms into itself under orthogonal rotation of axes. Thus, for instance, the equation $\delta_{ij} = l_{im} l_{jn} \delta_{mn}$ shows that the Kronecker delta is an isotropic tensor of the second order. Also for specifically *right-handed* transformations of orthogonal axes, (1.70) shows that the third-order pseudo-tensor

ε_{ijk} is isotropic. Scalars are also isotropic, but first-order tensors, which are vectors, are not isotropic since only those vectors parallel to the axis of rotation remain invariant under the rotation.

It is of some importance to examine the special case of **isotropic tensors of the fourth order** which feature in the theory of deformable media. We seek the most general fourth-order tensor of the form c_{ijkl} which is isotropic under the rotations specified by (1.78). Its appropriate transformation law becomes

$$c_{ijkl} = a_{ir}a_{js}a_{kt}a_{ln}c_{rstn}, \tag{1.81}$$

involving four successive finite rotations of the kind considered in the last section ($i, j, k, l; r, s, t, n = 1, 2, 3$).

When we rotate the axes through π about the x_3 axis, (1.79) shows that

$$a_{ij} = -\delta_{ij} + 2n_in_j.$$

For this rotation, $n_1 = 0 = n_2$, $n_3 = 1$ and the only non-zero components of a_{ij} are

$$a_{11} = -1, \ a_{22} = -1, \ a_{33} = 1.$$

Substituting into (1.81) gives $c_{ijkl} = -c_{ijkl}$, or $c_{ijkl} = 0$, in the following cases.

(1) Any three of the indices equal to 1 and the other to 3.
(2) Any three of the indices equal to 2 and the other to 3.
(3) Any two of the indices equal to 1, another equal to 2 and the fourth equal to 3.
(4) Any two of the indices equal to 2, another equal to 1 and the fourth equal to 3.

We obtain similar results on making rotations of π about the x_1 and x_2 axes. So for rotations through π about the three coordinate axes the only non-zero components that arise are those with four indices equal or equal in pairs.

Now consider a rotation of the axes through $\pi/2$ about the x_3 axes. Equation (1.79) shows that now

$$a_{ij} = n_in_j - \varepsilon_{ijk}n_k,$$

with $n_1 = 0 = n_2$, $n_3 = 1$. Then the only non-zero components are

$$a_{12} = -1, \ a_{21} = 1, \ a_{33} = 1.$$

Direct substitution into (1.81) shows that

$$\begin{aligned} &c_{1111} = c_{2222}, \\ &c_{1122} = c_{2211}, \quad c_{1133} = c_{2233}, \quad c_{3311} = c_{3322}, \\ &c_{1212} = c_{2121}, \quad c_{1313} = c_{2323}, \quad c_{3131} = c_{3232}, \\ &c_{1221} = c_{2112}, \quad c_{1331} = c_{2332}, \quad c_{3113} = c_{3223}. \end{aligned}$$

Similar results arise for rotations through $\pi/2$ about the x_1 and x_2 axes. We ca.. collect together these results in the following form. The subscripts i, j, k and l are all unequal and there is no summation. Throughout, $i, j, k, l = 1, 2, 3$.

$$\begin{aligned} &c_{iiii} = c_{jjjj}, \\ &c_{iijj} = c_{iikk} = c_{llij} = c_{llkk}, \\ &c_{ijij} = c_{ikik} = c_{ljlj} = c_{lklk}, \\ &c_{ijji} = c_{ikki} = c_{ljjl} = c_{lkkl}. \end{aligned} \tag{1.82}$$

All other components are zero. The most general solution of (1.82) is

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk} + \kappa\delta_{ijkl}, \tag{1.83}$$

where λ, μ, ν, κ are proper scalars and $\delta_{ijkl} = 1$ when all four indices are equal and otherwise zero.

If we now carry out a *small* rotation represented by

$$a_{ij} = \delta_{ij} + s_{ij},$$

where s_{ij} is a skew-symmetric tensor of the second order and is of the first order in small quantities, as defined by (1.80), then substituting into (1.81) and retaining only the first-order terms gives

$$\begin{aligned} c_{ijkl} &= (\delta_{ir}\delta_{js}\delta_{kt}\delta_{ln})c_{rstn} \\ &\quad + (s_{ir}\delta_{js}\delta_{kt}\delta_{ln} + \delta_{ir}s_{js}\delta_{kt}\delta_{ln} + \delta_{ir}\delta_{js}s_{kt}\delta_{ln} + \delta_{ir}\delta_{js}\delta_{kt}s_{ln})c_{rstn} + \cdots \\ &= c_{ijkl} + s_{ir}c_{rjkl} + s_{js}c_{iskl} + s_{kt}c_{ijtl} + s_{ln}c_{ijln} + \cdots, \end{aligned}$$

i.e.

$$s_{ir}c_{rjkl} + s_{js}c_{iskl} + s_{kt}c_{ijtl} + s_{ln}c_{ijkn} = 0.$$

Putting $i = 2$, $j = k = l = 1$ and using $s_{ij} = -s_{ji}$, we find that

$$s_{2r}c_{r111} + s_{1s}c_{2s11} + s_{1t}c_{21t1} + s_{1n}c_{211n} = 0$$

or

$$(s_{21}c_{1111} + s_{23}c_{3111}) + (s_{12}c_{2211} + s_{13}c_{2311}) + (s_{12}c_{2121} + s_{13}c_{2131}) + (s_{12}c_{2112} + s_{13}c_{2113}) = 0,$$

i.e.

$$c_{1111} = c_{2211} + c_{2121} + c_{2112},$$

on using $c_{3111} = c_{2311} = c_{2131} = c_{2113} = 0$. On substituting the appropriate forms for the c values given by (1.83) into the above expression for c_{1111}, we obtain

$$\lambda + \mu + \nu + \kappa = \lambda + \mu + \nu,$$

and so (1.83) reduces to

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \nu\delta_{il}\delta_{jk}. \tag{1.84}$$

This is the most general isotropic tensor of the fourth order.

In applications such as arise in studying the mechanics of deformable media, c_{ijkl} is symmetric in the pairs of indices (i, j); (k, l) and the components λ, μ, ν are all constants. When we use $c_{ijkl} = c_{ijlk}$ in association with (1.84), we obtain

$$(\mu - \nu)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) = 0.$$

The second factor is not generally zero. Putting $i = 1 = k$, $j = 2 = l$, this factor is unity. Hence, it is generally true that $\mu = \nu$ and (1.84) simplifies to the form

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}).$$

1.9 DYADICS

Let **b** and **c** be two vectors with components b_i and c_i, respectively. Their scalar product $\mathbf{b}\cdot\mathbf{c}$ is the inner product $b_i c_i$. The quantity **bc** is the indefinite product of the two vectors. It is referred to as a **dyadic**. Letting **D** denote this dyadic, we have

$$\mathbf{D} = D_{ij}\mathbf{e}_i\mathbf{e}_j = b_i c_j \mathbf{e}_i\mathbf{e}_j, \tag{1.86}$$

where $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a right-handed triad of unit vectors. In (1.86) the summation convention on repeated subscripts is used. Thus, **D** involves *nine* scalar quantities D_{ij} which are the *components* of the dyadic **D**. Although their numerical values depend on the particular system of coordinates employed, the dyadic **D** has a significance which is independent of any coordinate system. Associated with the dyadic is the 3×3 matrix

$$[D_{ij}] = \begin{bmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{bmatrix} \tag{1.87}$$

and, from the definition of D_{ij}, the reader will see that the elements of the matrix of **D** are also the elements of a second-order tensor. The determinant of **D** is

$$\det \mathbf{D} = \begin{vmatrix} D_{11} & D_{12} & D_{13} \\ D_{21} & D_{22} & D_{23} \\ D_{31} & D_{32} & D_{33} \end{vmatrix}, \tag{1.88}$$

whose value can be shown to be an invariant, i.e. independent of the choice of coordinate system employed.

The **transpose** $\mathbf{D}^{\dagger}$ of **D** is the dyadic obtained by interchanging the order of the unit vectors. Thus

$$\mathbf{D}^{\dagger} = D_{ij}\mathbf{e}_j\mathbf{e}_i = D_{ji}\mathbf{e}_i\mathbf{e}_j, \tag{1.89}$$

remembering that i and j are merely dummy suffices.

A dyadic is said to be **symmetric** if $\mathbf{D}^{\dagger} = \mathbf{D}$, which is equivalent to

$$D_{ij} = D_{ji} \qquad (i, j = 1, 2, 3). \tag{1.90}$$

A symmetric dyadic thus possesses only *six* independent components and any symmetric dyadic **D** can be expressed as

$$\mathbf{D} = D_1\xi_1\xi_1 + D_2\xi_2\xi_2 + D_3\xi_3\xi_3, \tag{1.91}$$

where (ξ_1, ξ_2, ξ_3) are the three mutually perpendicular unit eigenvectors of the symmetric dyadic **D**. The three scalars D_1, D_2, D_3 are the eigenvectors of **D**. The reader will observe that the problem of finding the eigenvalues and eigenvectors of **D** is equivalent to finding the same quantities for the symmetric matrix $[D_{ij}]$ or, in other words, expressing $[D_{ij}]$ in diagonal form. A dyadic **D** is **skew symmetric** if

$$\mathbf{D}^{\dagger} = -\mathbf{D}. \tag{1.92}$$

Any dyadic can be uniquely expressed as the sum of a symmetric and skew-symmetric dyadic as follows:

$$\mathbf{D} = \tfrac{1}{2}(\mathbf{D} + \mathbf{D}^{\dagger}) + \tfrac{1}{2}(\mathbf{D} - \mathbf{D}^{\dagger}),$$

since the first term on the right-hand side is symmetric while the second term is skew symmetric.

A particularly important dyadic is the **idemfactor** or unit dyadic **I**, defined by

$$\mathbf{I} = \delta_{ij}\mathbf{e}_i\mathbf{e}_j, \tag{1.93}$$

states that

$$\int_S \mathrm{d}\mathbf{S}\cdot\mathbf{D} = \int_V \nabla\cdot\mathbf{D}\,\mathrm{d}V \tag{1.106}$$

where S is a closed surface bounding the volume V and $\mathrm{d}\mathbf{S}$ is the vector areal element of S whose direction points along the normal directed out of the volume V. This follows since

$$\int_V \nabla\cdot\mathbf{D}\,\mathrm{d}V = \int_V \mathbf{e}_k \frac{\partial}{\partial x_i} D_{ik}\,\mathrm{d}V$$
$$= \int \mathbf{e}_k \mathbf{e}_i D_{ik}\,\mathrm{d}S$$

using the divergence theorem for vectors applied to the vector $\mathbf{A} = D_{ik}\mathbf{e}_i$ with the unit normal to S given by $\mathbf{n} = l_i\mathbf{e}_i$. However,

$$\mathbf{e}_k l_i D_{ik} = \mathbf{n}\cdot\mathbf{D}$$

and thus

$$\int \mathrm{d}\mathbf{S}\cdot\mathbf{D} = \int \nabla\cdot\mathbf{D}\,\mathrm{d}V.$$

Since, in general, $\mathrm{d}\mathbf{S}\cdot\mathbf{D} \neq \mathbf{D}\cdot\mathrm{d}\mathbf{S}$, the correct ordering of the vectors and dyadic in the theorem must be adhered to.

PROBLEMS 1

(1) Establish the following vector identities:

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c});$$
$$(\mathbf{a}\times\mathbf{b})\times(\mathbf{c}\times\mathbf{d}) = [\mathbf{c},\mathbf{d},\mathbf{a}]\mathbf{b} - [\mathbf{b},\mathbf{c},\mathbf{d}]\mathbf{a} = [\mathbf{d},\mathbf{a},\mathbf{b}]\mathbf{c} - [\mathbf{a},\mathbf{b},\mathbf{c}]\mathbf{d}.$$

(2) Establish the following results:

$$\operatorname{curl}(\phi\mathbf{A}) = \phi \operatorname{curl}\mathbf{A} + (\operatorname{grad}\phi)\times\mathbf{A},$$
$$\operatorname{curl}(\mathbf{A}\times\mathbf{B}) = \mathbf{A}\operatorname{div}\mathbf{B} - \mathbf{B}\operatorname{div}\mathbf{A} + (\mathbf{B}\cdot\operatorname{grad})\mathbf{A} - (\mathbf{A}\cdot\operatorname{grad})\mathbf{B}.$$

Evaluate $\operatorname{curl}(\mathbf{a}\times\mathbf{r}/r^n)$, where $\mathbf{a}$ is a constant vector, $\mathbf{r} = (x, y, z)$ is the position vector of a point, and $r = |\mathbf{r}|$.

(3) Prove from first principles that

$$\iint \mathbf{n}\times\nabla V\,\mathrm{d}S = \oint V\,\mathrm{d}\mathbf{r},$$

where V is a single-valued differentiable scalar function of position, $\mathbf{n}\,\mathrm{d}S$ is a vector element of an open surface (assumed to be suitably simple) and $\mathrm{d}\mathbf{r}$ is a vector element of the closed curve bounding the surface.

(4) A given vector $\mathbf{u}$ is a continuous and differentiable function of position in a simply connected region D. Show that $\int \mathbf{u}\cdot\mathrm{d}\mathbf{s}$ along a path between any two points in D is independent of the path if, and only if, $\operatorname{curl}\mathbf{u} = \mathbf{0}$ everywhere in D. (Theorems quoted in the proofs should be stated clearly but need not be proved.)

If $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, determine which of the following vector functions satisfy the condition in the given domains.

(a) $r^n\mathbf{r}$ in the whole space, where $n > 0$, $r = |\mathbf{r}|$.
(b) $\mathbf{r}\times\mathbf{k}$ in the whole space.
(c) $\dfrac{\mathbf{r}\times\mathbf{k}}{x^2+y^2}$ in the region $x^2 + y^2 \geqslant 1$, $x > 0$.

Where appropriate, evaluate the line integral between the points whose position vectors are $\mathbf{a}$ and $\mathbf{b}$.

(5) (a) Assuming the divergence theorem, prove Green's theorem that

$$\iiint_\tau (\nabla U\cdot\nabla V + U\,\nabla^2 V)\,\mathrm{d}\tau = \iint_S U\frac{\partial V}{\partial n}\,\mathrm{d}S.$$

Given two single-valued functions of positions, U and V, whose second partial derivatives are continuous in a simply connected volume τ and on its boundary surface S, show that, if U and V are harmonic in τ, and if

$$\frac{\partial U}{\partial n} = \frac{\partial V}{\partial n} \text{ on } S,$$

then $U - V$ is constant in τ.

(b) Using spherical polar coordinates, find a harmonic function U, finite at the origin, such that

$$\frac{\partial U}{\partial r} = 1 - 3\cos^2\theta$$

on the spherical surface $r = 2$.

(6) The quantities ϕ and $\mathbf{A}$, respectively, are scalar and vector functions of position

within a simply connected volume V bounded by a closed surface S. Show that

$$\int_S \phi\, d\mathbf{S} = \int_V \operatorname{grad} \phi\, dV.$$

$$\int_S \mathbf{A} \times d\mathbf{S} = -\int_V \operatorname{curl} \mathbf{A}\, dV.$$

Show that, if $\mathbf{a}$ is any constant vector, and $\mathbf{r}$ the position vector of the element $d\mathbf{S}$, then

$$\int_S (\mathbf{r} \times \mathbf{a}) \times d\mathbf{S} = 2V\mathbf{a}.$$

(7) Prove that, if $\phi(x_1, x_2, x_3)$ has continuous first derivatives in a volume V bounded by a surface S, then

$$\int_V \frac{\partial \phi}{\partial x_j}\, d\tau = \int_S \phi\, dS_j,$$

where dS_j is the projection of the element of area dS on the plane $x_j = 0$. Deduce that

$$\int_V \operatorname{curl} \mathbf{A}\, d\tau = \int_S (\mathbf{n} \times \mathbf{A})\, dS,$$

where $\mathbf{A}$ is any vector with continuous first derivatives, and $\mathbf{n}$ is the unit vector along the outward normal to S.

By applying this result to a plane lamina of small uniform thickness, show that

$$\int_S (\mathbf{n} \cdot \operatorname{curl} \mathbf{A})\, dS = \int_s \mathbf{A} \cdot d\mathbf{s},$$

where the plane area S is bounded by the curve s.

(8) A given differentiable scalar function λ is positive in a domain D. Differentiable vector functions $\mathbf{u}, \mathbf{u}'$ satisfy $\nabla \cdot \mathbf{u}' = \nabla \cdot \mathbf{u}$ in D and $u_n' = u_n$ on S, the boundary of D. Show that $\nabla \times (\lambda \mathbf{u}) = \mathbf{0}$ is a sufficient condition that

$$\int_D \lambda \mathbf{u}' \cdot \mathbf{u}'\, dV \geqslant \int_D \lambda \mathbf{u} \cdot \mathbf{u}\, dV$$

for all $\mathbf{u}'$.

(9) Assuming the divergence theorem, and given that ϕ, ψ are twice differentiable functions of position in a volume τ and on its bounding surface S, deduce Green's theorem

$$\int_\tau (\phi \nabla^2 \psi - \psi \nabla^2 \phi)\, d\tau = \int_S (\phi \nabla \psi - \psi \nabla \phi) \cdot d\mathbf{S}.$$

A solution of the wave equation

$$\frac{\partial^2 V}{\partial t^2} - c^2 \nabla^2 V = 0$$

is of the form $\phi(r)f(t)$. Prove that, if $f(t)$ is sinusoidal, then ϕ satisfies the equation

$$\nabla^2 \phi + k^2 \phi = 0,$$

where k is a constant. Show that $[\cos(kr)]/r$ is a solution of this equation in any region not including the origin, and use Green's theorem to prove that, for every solution ϕ regular at the origin,

$$\phi(0) = -\frac{1}{4\pi} \int_S \left[\phi \frac{\partial}{\partial n} \left(\frac{\cos(kr)}{r} \right) - \frac{\cos(kr)}{r} \frac{\partial \phi}{\partial n} \right] dS$$

assuming that the origin is within S.

(10) Establish without assuming the divergence theorem the result

$$\int_S l\phi\, dS = \int_V \frac{\partial \phi}{\partial x}\, dV$$

where V is a volume enclosed by a surface S whose outward normal has direction cosines l, m, n and ϕ is a scalar function with continuous derivatives throughout V.

Hence deduce the volume integral transformations of the surface integrals

$$\int_S \mathbf{n}\phi\, dS, \quad \int_S \mathbf{n} \cdot \mathbf{q}\, dS, \quad \int_S \mathbf{n} \times \mathbf{q}\, dS,$$

where $\mathbf{n}$ is the unit vector along the outward normal.

Show that the position vector of the centroid of the volume V is given by

$$\frac{1}{2V} \int_S \mathbf{n} r^2\, dS.$$

and use this formula to find the position of the centre of mass of the uniform solid hemisphere $r \leqslant a$, $z \geqslant 0$.

(11) Show how to construct a triply orthogonal system of surfaces of revolution by taking cylindrical coordinates (ρ, ϕ, z) such that $\rho(\alpha, \beta) + iz(\alpha, \beta) = f(\alpha + i\beta)$, where α and β are real parameters. Show that the system defined by the vector

$$\mathbf{r} = (\cos\alpha \cosh\beta \cos\gamma,\ \cos\alpha \cosh\beta \sin\gamma,\ \sin\alpha \sinh\beta)$$

is of this kind.

Prove that, in these coordinates, the equation $\nabla^2 V = 0$ is

$$\cosh\beta \frac{\partial}{\partial\alpha}\left(\cos\alpha \frac{\partial V}{\partial\alpha}\right) + \cos\alpha \frac{\partial}{\partial\beta}\left(\cosh\beta \frac{\partial V}{\partial\beta}\right) + \frac{\cosh^2\beta - \cos^2\alpha}{\cos\alpha\cosh\beta}\frac{\partial^2 V}{\partial\gamma^2} = 0.$$

(12) Explain the meaning of the statements that u_i $(i = 1, 2)$ are components of a vector and t_{ik} $(i, k = 1, 2)$ are components of a tensor in two-dimensional Euclidean space. If u_i and v_i are vector components and t_{ik} tensor components, prove that $u_i v_i$ (with the summation convention) is a scalar, and $t_{ik} v_k$ are components of a vector.

If $\varepsilon_{11} = \varepsilon_{22} = 0$, $\varepsilon_{12} = -\varepsilon_{21} = 1$, prove that the components ε_{ik} transform as tensor components under rotations but not under reflections. Prove also that, if $\mathbf{u}$ and $\mathbf{v}$ are vectors, the object $[\mathbf{v}]$ with components $v_2, -v_1$ transforms under rotations as a vector, and the number $[\mathbf{uv}] = u_1 v_2 - u_2 v_1$ as a scalar.

(13) State the transformation property of the following.

(a) A vector.
(b) A tensor of second order.

If u_i $(i = 1, 2, 3)$ are components of a vector referred to a set of Cartesian axes Ox_1, Ox_2, Ox_3, show that

$$e_{ik} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right)$$

are components of a symmetric tensor. Show that the components of the tensor referred to any frame of reference may be written

$$e_{ji} = \tfrac{1}{2}[\hat{\mathbf{n}}_j \cdot (\hat{\mathbf{n}}_i \cdot \nabla)\mathbf{u} + \hat{\mathbf{n}}_i \cdot (\hat{\mathbf{n}}_j \cdot \nabla)\mathbf{u}]$$

where $\hat{\mathbf{n}}_j$ and $\hat{\mathbf{n}}_i$ are unit vectors in the frame.

(14) Define the alternating tensor ε_{ijk}, and prove that

$$\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}.$$

A solenoidal field $\mathbf{H}$ exerts a force $(\nabla \times \mathbf{H}) \times \mathbf{H}/4\pi$ on unit volume. Show that the total force on the volume V within a closed surface S is

$$\int_S T_{il} n_l \, dS,$$

where T_{il} is a certain symmetric tensor and n_l is the unit outward normal to the surface element dS. Show also that the total couple about a point O is

$$\int_S \varepsilon_{ijk} x_j T_{kl} n_l \, dS,$$

where x_j is the position vector relative to O of the element of surface $d\mathbf{S}$.

(15) The gradient of ϕ may be denoted by $\partial\phi/\partial\mathbf{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Thus

$$\frac{\partial}{\partial\mathbf{r}} = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}.$$

If $\mathbf{A}$ is a constant vector and $\mathbf{I}$ is the idemfactor, prove the following.

(a) $\dfrac{\partial\mathbf{r}}{\partial\mathbf{r}} = \mathbf{I}$.

(b) $\dfrac{\partial}{\partial\mathbf{r}}(\mathbf{A}\cdot\mathbf{r}) = \mathbf{A}$.

(c) $\dfrac{\partial\phi}{\partial x} = \dfrac{\partial\phi}{\partial\mathbf{r}} \cdot \dfrac{\partial\mathbf{r}}{\partial x}$.

(16) Letting $\boldsymbol{\xi} = \xi_1\mathbf{i} + \xi_2\mathbf{j} + \xi_3\mathbf{k}$ and $\boldsymbol{\eta} = \eta_1\mathbf{i} + \eta_2\mathbf{j} + \eta_3\mathbf{k}$, we may define

$$\frac{\partial}{\partial\boldsymbol{\xi}} = \mathbf{i}\frac{\partial}{\partial\xi_1} + \mathbf{j}\frac{\partial}{\partial\xi_2} + \mathbf{k}\frac{\partial}{\partial\xi_3},$$

with a similar definition for $\partial/\partial\boldsymbol{\eta}$. Show that, if $F = F(\boldsymbol{\xi}, \boldsymbol{\eta})$, with $\boldsymbol{\xi} = \boldsymbol{\xi}(x, y, z)$ and $\boldsymbol{\eta} = \boldsymbol{\eta}(x, y, z)$, then

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial\boldsymbol{\xi}} \cdot \frac{\partial\boldsymbol{\xi}}{\partial x} + \frac{\partial F}{\partial\boldsymbol{\eta}} \cdot \frac{\partial\boldsymbol{\eta}}{\partial x}.$$

(17) Prove the following.

(a) $\nabla \cdot [\mathbf{qq} - \tfrac{1}{2}\mathbf{q}^2\mathbf{I}] = \mathbf{q}(\nabla \cdot \mathbf{q}) - \mathbf{q} \times (\nabla \times \mathbf{q})$.
(b) $\nabla \cdot [\mathbf{r} \times (\mathbf{qq} - \tfrac{1}{2}\mathbf{q}^2\mathbf{I})] = [\mathbf{q}(\nabla \cdot \mathbf{q}) - \mathbf{q} \times (\nabla \times \mathbf{q})] \times \mathbf{r}$